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Second-order radio frequency kinetic theory revisited: Resolving inconsistency with conventional fluid theory

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The second-order velocity distribution function was calculated from the second-order rf kinetic theory [Jaeger *et al*., Phys. Plasmas **7**, 641 (2000)]. However, the nonresonant ponderomotive force in the radial direction derived from the theory is inconsistent with that from the fluid theory. The inconsistency arises from that the multiple-timescale-separation assumption fails when the second-order Vlasov equation is directly integrated along unperturbed particle orbits. A slowly ramped wave field including an adiabatic turn-on process is applied in the modified kinetic theory in this paper. Since this modification leads only to additional reactive/nonresonant response relevant with the secular resonant response from the previous kinetic theory, the correct nonresonant ponderomotive force can be obtained while all the resonant moments remain unchanged. © *2013 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4817812]

I. INTRODUCTION

The direct force exerted by rf waves on magnetized plasmas is undoubtedly a basic quantity for rf current/flow drive.[1] The wave deposits the momentum on resonant particles through collisionless damping and then provides resonant forces. In addition, the bulk plasma can experience the nonresonant ponderomotive force in an inhomogeneous rf field. Therefore, a rf kinetic theory might be required to calculate the resonant and the nonresonant interactions in a unified framework. Several kinetic theories based on the Vlasov equation (or collisionless Boltzmann equation) were developed to formulate these forces[2–4] and used to analyze the flows observed in experiments or simulations.[5–7]

A second-order rf kinetic theory developed by Jaeger, Berry, and Batchelor (JBB) is quite useful to calculate all the forces because the slowly varying part of the second-order velocity distribution function was explicitly solved.[2] From this theory, the poloidal and toroidal components of the force (i.e., those in the flux surface) were successively obtained.[8,9] The obtained poloidal resonant force has been used to interpret experimental results[5] and been applied in simulation works;[6] while the local parallel force was found to be completely resonant, which was used successfully to clear up a long-standing dispute on the matter of current drive by nonresonant force.[10] However, the radial force was not given correctly from the JBB solution. For example, in the cold-fluid limit, the force from their solution reduces to a nonresonant ponderomotive force as

$$F_{\rm nr} = -\frac{n_0 q^2}{4m\omega^2}\frac{\partial}{\partial x}|E_z|^2, \quad (1)$$

while the force from the fluid theory[11,12] is

$$F_{cf} = -\frac{n_0 q^2}{4m}\left(\frac{1}{\omega^2}\frac{\partial|E_z^2|}{\partial x}+\frac{1}{2(\omega-\Omega)^2}\frac{\partial|E_+^2|}{\partial x}+\frac{1}{2(\omega+\Omega)^2}\frac{\partial|E_-^2|}{\partial x}\right), \quad (2)$$

where $E_\pm = E_x \pm iE_y$, Ω is the Larmor cyclotron frequency, and the z-axis is parallel to the uniform, static equilibrium magnetic field. Then two questions are concerned in this paper. One is how to upgrade the theory to resolve this disagreement, and the other is whether the modification affects previous conclusions of resonant forces in the flux surface or other resonant moments.

In principle, the JBB theory is an approach of integrating the slowly varying (in time) part of the second-order Vlasov equation along unperturbed particle orbits. It is well known that the linear dispersion relation can be obtained by integrating the first-order Vlasov equation along unperturbed particle orbits. However, when one solves the second-order Vlasov equation as an initial value problem with the initial condition $f_2|_{t\le 0} = 0$, the incident wave field applied in the JBB work is actually equivalent to

$$\mathbf{E}_1(\mathbf{r},t) = \begin{cases} \widehat{\mathbf{E}}(\mathbf{r})\exp(-i\omega t) & t>0 \\ 0 & t\le 0 \end{cases}, \quad (3)$$

whose Fourier transform is

$$\int_{-\infty}^{\infty}\mathbf{E}_1(\mathbf{r},t)\exp(i\widehat{\omega}t)\mathrm{d}t = \widehat{\mathbf{E}}(\mathbf{r})\left[\frac{i}{(\widehat{\omega}-\omega)}+\pi\delta(\widehat{\omega}-\omega)\right]. \quad (4)$$

On the one hand, this setup gives a non-monochromatic spectrum. The spectrum has no influence on the linear dispersion relation, but may induce some nonphysical modes (e.g., local upper-hybrid modes described in Appendix A of Ref. 13). When the Vlasov equations are solved as initial

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value problems in theoretical analysis or particle simulations, these spurious modes could produce nonphysical response through the nonlinear self-coupling. On the other hand, the setup invalidates the multiple-timescale-separation assumption as follows. Equation (3) indicates that the evolution timescale T_{env} for the envelop evolution is much smaller than the oscillation period $2\pi/\omega$ at the initial time $t=0$. Then the time-averaging over the oscillation period diminishes the turn-on process of the envelop as well as the fast oscillation. The adiabatic turn-on of the wave field should be retained to obtain the nonresonant ponderomotive effect.[12] Thus, it is expected that the direct orbit integration will lead to the wrong ponderomotive force.

In this paper, we will apply a slowly ramp-up rf field as the driving field for the rf kinetic equation, which is

$$\mathbf{E}_1(\mathbf{r},t) = \begin{cases} \widehat{\mathbf{E}}(\mathbf{r})\exp(i\omega t)(1-\mathrm{e}^{-\beta t}), & t>0 \\ 0, & t\le 0, \end{cases} \tag{5}$$

where the timescale for the envelop evolution satisfies $T_{env}\sim 1/\beta \gg 2\pi/\omega$. This applied field make the timescale separation feasible (i.e., $\omega t \gg \beta t \gg 1$), and gives a roughly monochromatic spectrum. The specified form in Eq. (5) is not exclusive. For example, the ramp-up factor can be $(1-\mathrm{e}^{-\beta t})^2$ as that in Ref. 13 for particle-in-cell simulation, which can also separate the timescale and lead to the correct nonresonant response. After the modification, additional nonsecular reactive/nonresonant components of the second-order distribution function are obtained, which are closely related to the secular resonant component in the original solution by JBB. Furthermore, a correct nonresonant force consistent with the fluid theory is obtained. Meanwhile the resonant forces in the flux surface and other resonant moments are still the same as that from original JBB theory, since the resonant components do not change at all.

Another kinetic theory based on guiding-center formulation was developed by Myra and D'Ippolito (MD).[12] Due to the clear timescale separation of the second-order distribution into the gyro-averaged and the gyro-angle dependent components, the forces and the energy absorption rate were obtained even without the explicit solution of the second-order distribution function. Their work indicates that the secular response only exists in the guiding-center distribution function which may be affected by our modification of the wave field. Using a multiple-timescale analysis for the gyro-averaged Vlasov equation, the evolution equation for the diagonal terms of the pressure tensor was obtained, and then the nonresonant ponderomotive forces in the electrostatic (ES) limit and in the cold-fluid limit could be calculated (see Appendix B in Ref. 12). The explicit solution of f_2 given in this paper is consistent with their theory.

The rest of the paper is organized as follows. To avoid complexity, a simple case of ES wave in unmagnetized plasmas is introduced in Sec. II. The rf forces are derived from the fluid theory, the original JBB kinetic theory and the modified second-order kinetic theory, respectively. In Sec. III, the general case for electromagnetic (EM) wave in magnetized plasmas and the discussion with guiding-center formulation by MD are presented. Finally, a discussion and summary is given in Sec. IV.

II. ELECTROSTATIC WAVE IN UNMAGNETIZED PLASMAS

A. Kinetic formulation of rf force for electrostatic wave

Ignoring collisions on the rf timescale, the velocity distribution function can be obtained from the Vlasov equation

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f + \frac{q}{m}[\mathbf{E}_1 + v\times(\mathbf{B}_0+\mathbf{B}_1)]\cdot\nabla_{\mathbf{v}} f = 0. \tag{6}$$

The distribution function is expanded in powers of the perturbed electric fields as $f=f_0+f_1+f_2$, where f_0 is the equilibrium distribution function, f_1 is the linear response to the perturbed field, and f_2 is the slowly varying (in time) part of the second-order response. Following the JBB work we assume that the wave amplitude varies slowly along the x-axis. Then the rf force can be defined as the sum of quasi-linear electromagnetic force and nonlinear stress force, i.e.,

$$F_x \equiv F_{EM,x} - \nabla_x \cdot P_{xx}, \tag{7}$$

where $\mathbf{F}_{EM} \equiv \int q\langle[\mathbf{E}_1(\mathbf{r},t)+\mathbf{v}\times\mathbf{B}_1(\mathbf{r},t)]f_1(\mathbf{r},\mathbf{v},t)\rangle_t \mathrm{d}^3\mathbf{v}$ is the quasi-linear kinetic electromagnetic force and $\mathbf{P} \equiv m\int f_2(\mathbf{r},\mathbf{v},t)\mathbf{v}\mathbf{v}\mathrm{d}^3\mathbf{v}$ is the nonlinear kinetic stress. The time average for any nonlinear product AB, expressing in terms of their Fourier representations, is frequently abbreviated as

$$\langle AB\rangle_t = \sum_{k,k'}\frac{1}{4}\mathrm{e}^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}A^*{}_{k'}B_k + c.c. \tag{8}$$

For the sake of simplicity, we consider a problem of an ES wave propagating in unmagnetized plasmas (i.e., $\mathbf{B}_0=\mathbf{B}_1=0$) in this section. Here the wave vector is also oriented along the x-axis, i.e., $E_1(\mathbf{r},t)\propto E(x)\exp(ikx-i\omega t)$. With null magnetic field, the kinetic electromagnetic force reduces to quasi-linear electrostatic force, i.e., $\mathbf{F}_{ES} \equiv \int q\langle\mathbf{E}_1(\mathbf{r},t)f_1(\mathbf{r},\mathbf{v},t)\rangle_t\mathrm{d}^3\mathbf{v}$, and the complete nonlinear stress still requires the solution of the second-order distribution function.

B. Fluid theory

We first calculate the rf force in the cold-fluid limit from the fluid theory. In the cold-fluid limit, the quasi-linear electrostatic force becomes $\mathbf{F}_{ES}=\langle qn_1\mathbf{E}_1\rangle_t$ while the nonlinear stress reduces to the well-known Reynolds stress $\mathbf{P}=\langle n_0 m\mathbf{v}_1\mathbf{v}_1\rangle_t$. Using the momentum equation,

$$-i\omega m v_1 = qE, \tag{9}$$

and the continuity equation,

$$-i\omega n_1 = -n_0(\nabla + ik)v_1, \tag{10}$$

we can obtain the quasi-linear electrostatic force as

$$F_{ES} = \frac{n_0 q^2}{4m\omega^2}\nabla|E|^2, \tag{11}$$

and the nonlinear stress force as

$$-\nabla_x P_{xx} = -\frac{n_0 q^2}{2m\omega^2}\nabla|E|^2. \quad (12)$$

Thus, in the cold-fluid limit, the rf force due to an inhomogeneous ES wave is

$$F_{cf,ES} = -\frac{n_0 q^2}{4m\omega^2}\nabla|E|^2. \quad (13)$$

C. Original second-order kinetic theory

Now, we calculate the rf force induced by the ES wave through the kinetic approach of integrating along unperturbed particle orbits with the same initial wave field applied as in the original JBB kinetic theory. Although the ES case has never been considered in Ref. 2, for simplicity, we may refer to this analysis as the original JBB result in the ES limit. The perturbed electric field of the wave is assumed as $E_1(x,t) \propto E(x/L)\exp(ikx - i\omega t)$, where $E(x/L)$ represents the slowly varying amplitude in real space and hereafter L is the wave-envelope scale length. For the perturbation calculation, we use the slowly varying approximation, i.e., $vt\nabla E(x/L)/E(x/L) \ll 1$. The force will be calculated to first order in vt/L.

The first-order Vlasov equation is solved for f_1 by integrating along unperturbed particle orbits

$$f_1(x,v,t) = -\frac{q}{m}\int_0^t \mathrm{d}t' E_1(x',t')\partial_v f_0(x',v') + f_1(x-vt,v,0). \quad (14)$$

It yields directly

$$f_1 = -\frac{q}{m}\mathrm{e}^{ikx-i\omega t}\left[\frac{i}{\bar\omega}(1-\mathrm{e}^{i\bar\omega t})E + \frac{1-\mathrm{e}^{i\bar\omega t}+i\bar\omega t}{\bar\omega^2}v\nabla E\right]\partial_v f_0 + f_1(x-vt,v,0), \quad (15)$$

where $\bar\omega \equiv \omega - kv$. As customary the initial terms of (15) [i.e., terms like $\exp(-ikvt)$] can be dropped by appealing to phase-mixing arguments[14] for $t \gg (k\bar v)^{-1}$. Therefore, in the JBB work, only the forced oscillation is retained as that in Stix's textbook,[15] i.e.,

$$f_1 = -\frac{q}{m}\mathrm{e}^{ikx-i\omega t}\left[\frac{i}{\bar\omega}E + \frac{(1+i\bar\omega t)v}{\bar\omega^2}\nabla E\right]\partial_v f_0. \quad (16)$$

Using Eq. (16) [instead of Eq. (15)], we obtain the quasi-linear electrostatic force as

$$F_{ES} = -\frac{q^2}{2m}\mathrm{Re}\int \mathrm{d}v\left(\frac{i}{\bar\omega}|E^2| + \frac{(1+i\bar\omega t)v}{2\bar\omega^2}\nabla|E^2|\right)\partial_v f_0. \quad (17)$$

With the argument of causality, we use the Sokhotski–Plemelj theorem

$$\lim_{\alpha\to 0}\frac{1}{\bar\omega + i\alpha} = \mathrm{P}\left(\frac{1}{\bar\omega}\right) - i\pi\delta(\bar\omega), \quad (18)$$

to divide the force into two parts as

$$F_{ES} = -\frac{q^2}{2m}\int \mathrm{d}v\pi\delta(\bar\omega)\left(|E^2| + \frac{tv}{2}\nabla|E^2|\right)\partial_v f_0 - \frac{q^2}{4m}\int \mathrm{d}v\mathrm{P}\left(\frac{1}{\bar\omega^2}\right)v\nabla|E^2|\partial_v f_0. \quad (19)$$

The first part, proportional to $\delta(\bar\omega)$, is the resonant force with the Landau damping while the other part is nonresonant. In the cold-fluid limit, $\bar\omega \approx \omega$ and the nonresonant part becomes $q^2/(4m\omega^2)\nabla|E^2|$, which is consistent with Eq. (11), the quasi-linear electrostatic force from the fluid theory.

The second-order distribution function is required to zero order in $(vt/L)^{-1}$ to obtain the nonlinear stress force to first order. The second-order Vlasov equation is

$$\partial_t f_2 + v\partial_x f_2 = -\frac{q}{m}\langle E_1(\mathbf{r},t)\partial_v f_1\rangle_t, \quad (20)$$

where f_1 is given in Eq. (16). Then Eq. (20) is solved by again integrating along unperturbed particle orbits with the initial condition $f_2|_{t\le 0} = 0$. The initial condition implies that the perturbed field before the initial time vanishes even without any explicit statement; and the multiple-timescale separation becomes unavailable while the equivalent wave field is not monochromatic. The linear response f_1 from Eq. (16) is still monochromatic because the initial information was dropped according to phase-mixing arguments. Therefore, in essence, the first-order Vlasov equation was solved as an eigenvalue problem. It is more natural to solve the first- and the second-order Vlasov equations as the same kind problem. These equations will both be solved as initial value problems in Subsection II D.

Returning to the orbit integral of Eq. (20), it yields

$$f_{2,0} = \frac{q^2}{2m^2}\mathrm{Re}|E^2|\partial_v\left(\frac{it}{\bar\omega}\partial_v f_0\right) = \frac{t\pi q^2}{2m^2}|E^2|\partial_v[\delta(\bar\omega)\partial_v f_0], \quad (21)$$

where the subscript 0 in $f_{2,0}$ means f_2 is retained to zero order in $(vt/L)^{-1}$, and the Sokhotski–Plemelj theorem is used at the second equal sign. Then the nonlinear stress force is obtained as

$$-\nabla_x\cdot P_{xx} = \frac{\pi q^2}{m}t\nabla|E^2|\int \mathrm{d}vv\delta(\bar\omega)\partial_v f_0, \quad (22)$$

which contains only the resonant force and is inconsistent with Eq. (12) in the cold-fluid limit. As a result, the incompleteness of nonresonant response leads to a wrong description of the whole radial ponderomotive force from the original JBB kinetic theory.

D. Upgraded second-order kinetic theory

We now calculate the radial force from the upgraded second-order kinetic theory by replacing the applied rf field with that of Eq. (5). As mentioned previously, this modification make the time-separation available and will lead to the correct reactive response as shown below. However, it is not clear whether the resonant response would change while the perturbed field changes. If we use the Sokhotski–Plemelj theorem again to expand terms like $1/(\bar{\omega} - i\beta)$, it yields

$$\lim_{\alpha\to 0}\frac{1}{\bar{\omega}+i\alpha-i\beta} = P\left(\frac{1}{\bar{\omega}-i\beta}\right) - 2i\pi\delta(\bar{\omega}-i\beta). \tag{23}$$

The first part on the right-hand side is a complex function instead of a pure real one as that in Eq. (18), and the resonant factor $\delta(\bar{\omega})$ does not appear explicitly. So it is hard to compare the modified result with that from the original JBB theory. Instead of using the Sokhotski-Plemelj theorem, we retain the initial terms in both f_1 and f_2, and then obtain the resonant factor by using the identity, $\lim_{t\to 0}\sin(\bar{\omega}t)/\bar{\omega} = \pi\delta(\bar{\omega})$. This method is the same as that used in Friedberg's textbook for discussions on collisionless damping from particle motions.[16]

Only the nonlinear stress force needs to be modified, so henceforth the distribution function is retained to zero order in $(vt/L)^{-1}$. With some direct calculations, we obtain the linear response as

$$f_{1,0} = g_1(0) - g_1(\beta), \tag{24}$$

with

$$g_1(s) \equiv -\frac{q}{m}\mathrm{e}^{ikx-i\omega t-st}(1-\mathrm{e}^{i\bar{\omega}t+st})\left[\frac{i}{\bar{\omega}-is}E\right]\partial_v f_0. \tag{25}$$

Substituting Eqs. (24) and (25) into Eq. (20) and integrating again along unperturbed particle orbits, the modified second-order distribution function is obtained as

$$f_{2,0} = f_{2,0}^{c} + f_{2,0}^{init}, \tag{26}$$

with

$$f_{2,0}^{c} \equiv g_2^c(0,0) - g_2^c(\beta,0) - g_2^c(0,\beta) + g_2^c(\beta,\beta) \tag{27}$$

and

$$f_{2,0}^{init} \equiv -\frac{q^2}{2m^2}\mathrm{Re}|E|^2\partial_v(\Gamma\Lambda\partial_v f_0), \tag{28}$$

where

$$g_2^c(r,s) \equiv \frac{q^2}{2m^2}\mathrm{Re}\frac{1-\mathrm{e}^{-rt-st}}{r+s}|E|^2\partial_v\left[\frac{i}{\bar{\omega}-is}\partial_v f_0\right], \tag{29}$$

$$\Gamma \equiv \frac{\mathrm{e}^{i\bar{\omega}t}-1}{\bar{\omega}} - \frac{\mathrm{e}^{i\bar{\omega}t-\beta t}-1}{\bar{\omega}+i\beta}, \tag{30}$$

$$\Lambda \equiv \frac{1}{\bar{\omega}} - \frac{1}{\bar{\omega}-i\beta}, \tag{31}$$

and

$$\left.\frac{1-\mathrm{e}^{-rt-st}}{r+s}\right|_{r+s=0} \equiv t. \tag{32}$$

Using the identity, $\lim_{t\to 0}\sin\bar{\omega}t/\bar{\omega} = \pi\delta(\bar{\omega})$, as well as

$$\frac{\cos\bar{\omega}t-1}{\bar{\omega}^2} = -\frac{\partial}{\partial\bar{\omega}}\frac{\cos\bar{\omega}t-1}{\bar{\omega}} - \frac{t\sin\bar{\omega}t}{\bar{\omega}}, \tag{33}$$

$f_{2,0}$ can be expressed as

$$f_{2,0} = \frac{q^2}{2m^2}\mathrm{Re}|E^2|\partial_v\{[t\pi\delta(\bar{\omega}) - K_{nr,0}]\partial_v f_0\}, \tag{34}$$

where the additional nonresonant component is obtained as

$$K_{nr,0} \equiv \mathrm{Re}\frac{i}{2\beta(\bar{\omega}-i\beta)} - \frac{\beta^2}{\bar{\omega}^2(\bar{\omega}^2+\beta^2)}. \tag{35}$$

Considering the multiple-timescale assumption ($\omega t \gg \beta t \gg 1$) and expanding to the first order in β, the nonresonant component becomes

$$K_{nr,0} = -\frac{1}{2\bar{\omega}^2}. \tag{36}$$

After comparing Eq. (34) with Eq. (21) we can conclude that the modification of the applied wave field only leads to an additional reactive/nonresonant term, i.e., the second term in the bracket of Eq. (34). Furthermore, this reactive component in the distribution function only results in additional reactive components of the moments, e.g., density, particle flux, energy, and so on. This additional component is nonsecular so that the time derivatives of those moments and the moment equations remain unchanged. Since the resonant response stays unchanged, the resonant moments (including the resonant force) from the previous kinetic theory are correct even without the modification of the applied wave field.

Now, we consider the nonresonant force in the cold-fluid limit, i.e., $|kv| \ll \omega$, by which the resonant component can be neglected. In consideration of $\beta/\omega \ll 1$, Eq. (36) can be used. With these approximations, the nonlinear stress force becomes

$$-\nabla_x\cdot P_{xx} = -\frac{n_0q^2}{2m\omega^2}\nabla|E^2|. \tag{37}$$

This stress force is as same as that from the fluid theory, Eq. (12). The quasi-linear electrostatic force can also be obtained and is the same as the previous kinetic result in Eq. (19), which is consistent with the fluid force in the cold-fluid limit. Therefore, the total radial nonresonant forces from the kinetic and the fluid theories become consistent with each other.

III. ELECTROMAGNETIC WAVES IN MAGNETIZED PLASMAS

A. Modified JBB theory

In this section, we present a general modification to the JBB theory in the case of EM waves in magnetized plasmas. The process is quite similar to the ES wave case shown in the last section. Generally, the tokamak plasma is modeled as a perpendicularly stratified, one-dimensional slab plasma where x, y, and z refer to radial, poloidal, and toroidal coordinates, respectively; and the wave amplitude varies slowly along the radial direction.[2]

We now consider the modified second-order kinetic theory. Replacing the wave field by that defined in Eq. (5) and solving the first-order Vlasov equation as an initial value problem with $f_1|_{t\le 0}=0$, we obtain the linear response as

$$f_1 = -\frac{2qf_M}{m\alpha^2}\int dk_x e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}e^{i\lambda\sin(\varphi-\theta)}\sum_l e^{il(\theta-\varphi)}\mathbf{H}_l\cdot\mathbf{E} \times\frac{1}{i}\left[\frac{1}{\bar\omega_l}(1-e^{i\bar\omega_l t})-\frac{1}{\bar\omega_l-i\beta}(e^{-\beta t}-e^{i\bar\omega_l t})\right], \quad (38)$$

where $\bar\omega_l\equiv\omega-l\Omega-k_z v_{th}$ and the other notations are the same as those in Ref. 2. Compared with that in the JBB work, f_1 here contains extra terms with β and the initial response $\exp(i\bar\omega_l t)$. Substituting Eqs. (5) and (38) into the second-order Vlasov equation and following the tedious calculation similar to the JBB method, we can obtain the modified second-order distribution. We omit the lengthy details and directly discuss the difference due to the modification. Compared with the solution in the JBB work, only the integration K_m in f_2 changes here. In the original JBB work, $K_m/\bar\omega_l$ defined at Eq. (32) in Ref. 2 was evaluated as

$$\frac{1}{\bar\omega_l}K_m \equiv \frac{1}{\bar\omega_l}\lim_{\gamma\to 0}\int_{-t}^{0} d\tau e^{\gamma\tau}e^{i(l-l'+m)\Omega\tau} = \begin{cases} P\left(\dfrac{t}{\bar\omega_l}\right)-i\pi t\delta(\bar\omega_l), & l-l'+m=0\\ P\left(\dfrac{1}{i(l-l'+m)\Omega}\dfrac{1}{\bar\omega_l}\right)-\dfrac{\pi\delta(\bar\omega_l)}{(l-l'+m)\Omega}, & l-l'+m\neq 0.\end{cases} \quad (39)$$

In this work, $K_m/\bar\omega_l$ becomes

$$\left(\frac{1}{\bar\omega_l}K_m\right)_\beta = \begin{cases} P\left(\dfrac{t}{\bar\omega_l}\right)-it\pi\delta(\bar\omega_l)+iP(K_{\mathrm{nr},l}), & l-l'+m=0\\ P\left(\dfrac{1}{i(l-l'+m)\Omega\bar\omega_l}\right)-\dfrac{\pi\delta(\bar\omega_l)}{(l-l'+m)\Omega}, & l-l'+m\neq 0\end{cases}, \quad (40)$$

where

$$K_{\mathrm{nr},l}\equiv \mathrm{Re}\frac{i}{2\beta}\frac{1}{\bar\omega_l-i\beta}-\frac{\beta^2}{\bar\omega_l^2(\bar\omega_l^2+\beta^2)}, \quad (41)$$

and expanding to the first order in β

$$K_{\mathrm{nr},l} = -\frac{1}{2\bar\omega_l^2}. \quad (42)$$

It should be noted that fast damping terms like $\exp(-\beta t)$ and fast oscillation terms like $\exp(i\Omega t)$ are neglected. In view of Eqs. (40)–(42), the modification only leads to an additional reactive component in the second-order distribution function as same as that for the ES case in the last section. It also means that all the resonant moments stay unchanged.

Then the modified forces can be obtained just after considering the modification to the secular terms from the JBB work. In the Appendix, the secular components of f_2 is found to be resonant, i.e., $P(t/\bar\omega_l)$ vanishes in the final form of f_2. Thus, for calculations only to the first order in β, we can replace $-it\pi\delta(\bar\omega_l)$ in the JBB work with $-it\pi\delta(\bar\omega_l)$ $-i/(2\bar\omega_l^2)$ to obtain the modified forces. Then the nonresonant radial force in the cold-fluid limit becomes

$$F_{\mathrm{nr}} = -\frac{n_0 q^2}{4m}\frac{1}{\omega^2}\frac{\partial}{\partial x}|E_z|^2+(F_a)_{T\to 0}, \quad (43)$$

where the additional force after the modification is

$$(F_a)_{T\to 0} = -\frac{n_0 q^2}{8m}\left(\frac{1}{(\omega-\Omega)^2}\frac{\partial|E_+^2|}{\partial x}+\frac{1}{(\omega+\Omega)^2}\frac{\partial|E_-^2|}{\partial x}\right). \quad (44)$$

The nonresonant force is now consistent with Eq. (2) from the fluid theory. Furthermore, the radial momentum equation can be obtained as

$$\frac{\partial}{\partial t}(mnU_x)-\Omega mnU_y = F_x, \quad (45)$$

where nU_x and nU_y represent the radial and the poloidal flux, respectively. Here the whole radial rf force is

$$F_x = \frac{1}{2}\mathrm{Re}\sum_l \mathbf{E}\cdot\left[\frac{\bar k_x}{\omega}W_l - t\partial_x\left(\frac{l\Omega}{\omega}W_l\right)\right]\cdot\mathbf{E}+F_{\mathrm{nr}}, \quad (46)$$

where W_l is the W matrix (defined in Refs. 3 and 8) related to the local energy absorption rate due to resonant interaction and F_{nr} represents the nonresonant force. The first term in the brackets, $(\bar k_x/\omega)W_l$, represents the resonant momentum absorption, while the second term $t\partial_x(W_l l\Omega/\omega)$ represents approximately the gradient of the increasing perpendicular pressure by resonant heating.

As mentioned above, only the secular terms in the original JBB work change through the modification. Since the secular terms are cancelled naturally for the calculation for the poloidal and the parallel forces,[8,9] those forces are still valid even without the modification. It is also expected from the derivation from the guiding-center formulation in Ref. 3 or from their physical picture described in Ref. 9.

B. Comparison with the guiding-center formulation

The guiding-center formulation of the rf kinetic theory developed by Myra *et al*. can produce all the moments needed. In the MD theory, the second order distribution function is separated into the gyro-averaged and the gyro-angle-dependent components, i.e., $f_2 = \langle f^{(2)}\rangle_\phi + \tilde{f}^{(2)}$. The index (2) is lifted to follow the MD's notation and to emphasize that those function are defined in guiding-center coordinate $(\mathbf{R}, v_\perp, v_{||}, \phi)$ not the lab coordinate $(\mathbf{r}, \mathbf{v})$. The evolutionary equation for the gyro-angle-dependent component is

$$-\Omega\frac{\partial \tilde{f}^{(2)}}{\partial\phi} = -\langle \mathbf{a}_1\cdot\nabla_v f_1\rangle_t + \langle \mathbf{a}_1\cdot\nabla_v f_1\rangle_{\phi,t}, \tag{47}$$

and that gyro-averaged kinetic equation is

$$\left(\frac{\partial}{\partial t} + v_{||}\nabla_{||}\right)\langle f^{(2)}\rangle_{\phi,t} = -\langle \mathbf{a}_1\cdot\nabla_v f_1\rangle_{\phi,t}, \tag{48}$$

where $\mathbf{a}_1 \equiv (q/m)(\mathbf{E}_1 + \mathbf{v}\times\mathbf{B}_1)$. In Ref. 3, the resonant force in flux surface to the first order in ρ/L was given through an ingenious technology without the explicit solution for these equations while the correct nonresonant force in ES limit was given in Appendix B in Ref. 12.

It is apparent that only the guiding-center distribution function $\langle f^{(2)}\rangle_{\phi,t}$ contains the secular response. If the explicit expression of f_2 is required (e.g., for benchmark with PIC simulation,[7] Vlasov simulation,[17] and drift kinetic simulation[18]), Eq. (48) would be integrated and then the same problem like that in JBB theory may be encountered. To compare the solution with the modified JBB result, the parallel convective term will be neglected below. The same correction method presented in Sec. II can be applied here to obtained $\langle f^{(2)}\rangle_{\phi,t}$, which equals to the corresponding components in the modified JBB solution. The obtained $\langle f^{(2)}\rangle_{\phi,t}$ is presented in the Appendix.

In the MD theory, the resonant forces in flux surface were calculated only with f_1 and $\tilde{f}^{(2)}$. Noting the timescale-separation assumption, i.e., $\partial/\partial t \ll \Omega$ and $v_{||}\nabla_{||} \ll \Omega$, Eq. (47) is enough for evaluating $\tilde{f}^{(2)}$. The assumption is consistent with the complete evolution equation for $\tilde{f}^{(2)}$ solved with the modified wave field of Eq. (5). Thus, those resonant forces from the MD theory stay unchanged even by using a slowly ramped field.

Now, we consider the method used in MD theory to obtain the reactive part of $\langle f^{(2)}\rangle_{\phi,t}$ or the relevant reactive part of stress tensor.[12] In brief, the right hand side of the gyro-kinetic equation should be expressed as time derivatives of reactive quantities considering "an adiabatic turn-on process in the past" (see Appendix B in Ref. 12). The slowing varying field can also be thought as a damping or growing wave with a complex frequency $\omega_c = \omega + i\gamma$ equivalently. In conventional quasilinear theory,[15] the resonant factor $1/\bar{\omega}_l$ is expanded as follows:

$$\frac{1}{\omega_c - l\Omega - k_z v_z} \approx P\left(\frac{1}{\bar{\omega}_l}\right) - i\pi\delta(\bar{\omega}_l) + i\gamma\frac{\partial}{\partial\omega}P\left(\frac{1}{\bar{\omega}_l}\right), \tag{49}$$

where the third term at the right hand side represents the reactive response. Noting that damping/growing rate can be replaced by a time derivative, i.e., $\gamma = (\partial/\partial t)/2$, then the reactive response can be re-expressed as $-iP(1/2\bar{\omega}_l{}^2)\partial_t$. The time derivative is now applied on second-order quantities, e.g., the square of field amplitude. After this replacement of the resonant factor in the right hand side of Eq. (48), one can directly get the reactive response related to the previous secular resonant response. The result is the same as that from the orbit integration with the modified applied field, which results in a secular response and an additional reactive component $\mathrm{Re}[P(K_{\mathrm{nr},l})] \approx -1/(2\bar{\omega}_l{}^2)$. In retrospect, the reactive response related to the secular response can be obtained, if the adiabatic turn-on process of the wave is included in the derivation either by using an explicitly form of slowly ramped wave field or by just assuming a time-varying wave amplitude as same as that in the conventional quasilinear theory or in the MD theory.

IV. DISCUSSION AND SUMMARY

There are some other kinetic theories on ponderomotive forces, e.g., the Lie transform approach by Cary and Kaufman,[19] in which the definition of ponderomotive force was equivalent to the single particle ponderomotive force excluding the Reynolds stress and the polarization stress. The difference between the fluid ponderomotive force and the single particle ponderomotive force was discussed in Ref. 11. For calculations of fluxes driven by rf waves, the nonlinear stress should be included.[1,2]

Although the nonresonant ponderomotive force cannot directly drive the poloidal or the toroidal flows, it can modify the radial electric field through the radial momentum balance.[1] It can also balance the Lorentz force due to the poloidal flow. Recently, a particle-in-cell simulation showed that nonlinear parametric decay of rf waves near a lower-hybrid-resonant layer could induce a large nonresonant ponderomotive force.[7] That force would lead to a strong poloidal diamagnetic flow which could not be effectively damped by classical or neoclassical viscosity. Correct nonresonant and resonant forces from a unified kinetic framework might be required to deal with these highly nonlinear phenomena. When the wave amplitude is strong enough, bulk particles are subject to orbit instability and then diffuse in position space.[20] More advanced tools, such as renormalization, are required to calculate the nonlinear forces near cyclotron resonance, however, beyond the scope of the present analysis.

In summary, we have revisited the nonlinear rf kinetic theory and resolved the incongruity between the nonresonant ponderomotive force from the second-order rf kinetic theory and the conventional fluid ponderomotive force. When the second-order Vlasov equation is treated as an initial value problem, the preset wave field should slowly ramp up with a rate smaller than the oscillation frequency to make the multiple-timescale separation feasible; otherwise, it leads to a wrong nonresonant force. This modification only leads to an additional reactive/nonresonant component of the second-order distribution function. Then the correct nonresonant ponderomotive force is obtained, while the resonant moments of the distribution function are the same as that from previous kinetic theory. In hindsight, when using the approach of integrating along unperturbed particle orbits to solve high order Vlasov equations, the applied EM field should be carefully selected to reproduce the rising process of the wave and the correct reactive response.

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APPENDIX: SECOND-ORDER DISTRIBUTION ON GUIDING-CENTER POSITION

Using the wave field of Eq. (3) and integrating the gyro-averaged kinetic equation (48) along unperturbed guiding-center orbits, the secular response of guiding-center distribution is obtained as

$$\begin{aligned}\langle f^{(2,s)}\rangle_{\phi,t} = &-\frac{q^2}{m^2\alpha^2}\sum_{k_x^R,k_x^L} \mathrm{e}^{i(k_x^R-k_x^L)R_x}\sum_l \mathrm{e}^{il(\theta_R-\theta_L)} \\ &\times\left\{\frac{1}{2}\left(\frac{1}{v_\perp}+\frac{\partial}{\partial v_\perp}\right)(A_+^*\mathrm{e}^{i\theta_L}\hat{J}_{l-1}+A_-^*\mathrm{e}^{-i\theta_L}\hat{J}_{l+1})\right. \\ &+\frac{k_z v_\perp}{2\omega}\frac{\partial}{\partial v_z}(E_+^*\mathrm{e}^{i\theta_L}\hat{J}_{l-1}+E_-^*\mathrm{e}^{-i\theta_L}\hat{J}_{l+1}) \\ &+\frac{\partial}{\partial v_z}E_z^*\left(1-\frac{l\Omega}{\omega}\right)\hat{J}_l \\ &\left.+\frac{i\delta k_x}{2\Omega}[i(A_+^*-A_-^*)\hat{J}_l - B^*(\mathrm{e}^{i\theta_L}\hat{J}_{l-1}+\mathrm{e}^{-i\theta_L}\hat{J}_{l+1})]\right\},\end{aligned} \quad \text{(A1)}$$

where $\hat{J}_m = if_M \mathrm{J}_m(\lambda_L)\mathbf{H}\cdot\mathbf{E}_R t/\bar{\omega}_l$, $A_\pm = E_\pm(1-k_z v_z/\omega) + E_z v_z k_\pm/\omega$, $B = (E_y k_x - E_x k_y)v_\perp/2\omega$, and

$$\begin{aligned}\mathbf{H}\cdot\mathbf{E}_R \equiv &\frac{1}{2}E_+ v_\perp \mathrm{J}_{l-1}(\lambda_R)e^{-i\theta_R}+\frac{1}{2}E_- v_\perp \mathrm{J}_{l+1}(\lambda_R)e^{i\theta_R} \\ &+E_z v_z \mathrm{J}_l(\lambda_R).\end{aligned} \quad \text{(A2)}$$

Equation (A1) just equals to the secular part of the original JBB result.[2] Then two conclusions can be drawn here. The first is that only the applied field of gyro-averaged kinetic equation needs modification to obtained the correct reactive component related to the secular component. The second is that the possible secular flux in the perpendicular plane is due to diamagnetic effect but not the drift of guiding-centers. The secular component of diamagnetic poloidal drift can be obtained as follows:

$$\int \mathrm{d}^3\mathbf{v}v_y\left(1+\frac{v_y}{\Omega}\partial_x\right)\langle f^{(2)}\rangle_{\phi,t} = \int \mathrm{d}^3\mathbf{v}\frac{v_\perp^2}{2\Omega}\partial_x\langle f^{(2)}\rangle_{\phi,t}, \quad \text{(A3)}$$

where the gradient comes from the transformation between the lab coordinate and the guiding-center coordinate.

Furthermore, $\langle f^{(2,s)}\rangle_{\phi,t}$ can be proved to be resonant to the first order in ρ/L. The process of proof includes adding or subtracting $n\Omega$ to cancel the resonant denominator and then using Graf's formula[21] to prove that the nonresonant terms are higher order small. Therefore, the secular resonant factor $t/\bar{\omega}_l$ in Eq. (A1) can be replaced by $-it\pi\delta(\bar{\omega}_l)$. The reactive components of $\langle f^{(2)}\rangle_{\phi,t}$ can be obtained just by replacing the secular resonant factor by $-i/(2\bar{\omega}_l^2)$ as that in the correction for the JBB result.

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