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ORIGINAL PAPER

Inductance of Bitter Coil with Rectangular Cross-section

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Abstract The self-inductance of Bitter coil and mutual inductance between coaxial Bitter coils with rectangular cross-section using semi-analytical expressions based on two integrations were introduced. The current density of the Bitter coil in radial direction is inversely proportional to its radius. The obtained expressions can be implemented by Gauss integration method with FORTRAN programming. We confirm the validity of inductance results by comparing them with finite filament method and finite element method. The inductance values computed by three methods are in excellent agreement. The derived expressions of inductance of Bitter coils with rectangular cross-section allow a low computational time compared with finite filament method to a specific accuracy. The derived mutual inductance expressions can be used to accurately calculate the axial force between coaxial Bitter coils with mutual inductance gradient method.

Keywords Bitter coil · Inductance · Storage energy

1 Introduction

Steady-state high magnetic field over 20 T can be generated by a water-cooled magnet with Bitter coils or by hybrid magnet combined with an inner water-cooled magnet and an outsert superconducting magnet. During a water-cooled magnet trip, the induced current in the superconducting outsert coil changes as a function of decay time constant of the water-cooled magnet. The decay time constant is determined by the self-inductance and the resistance of the water-cooled magnet. Thus, an accurate estimation of the inductance of Bitter coil is of primary importance in the hybrid magnet design. For a water-cooled magnet combined with some Bitter coils, there exists a large electromagnetic force in an axial direction. The electromagnetic force between Bitter coils is proportional to their mutual inductance gradient [1]. To optimize the support structure of the Bitter coil, it is vital to have accurate evaluation of the mutual inductance between Bitter coils.

Many early contributions on the self-inductance and mutual inductance of solenoid coils with rectangular cross-section are mainly concerned on the coils with uniform current density distribution [2–7]. For Bitter coils with copper disk, we neglect the variations of current density distribution caused by temperature distribution and cooling hole of Bitter coil. The current density distribution of a Bitter coil in radial direction, however, is inversely proportional to the radius of the coil [1]. In this study, we derived the expressions of self-inductance of Bitter coil and mutual inductance between Bitter coils based on two integrations. These expressions can be integrated with Gauss integration method by using FORTRAN programming. The results obtained by this method can be compared with the results by filament method and by finite element method with ANSYS software [4]. The filament method is the using of Maxwell's coils where coils are subdivided into filamentary coils [4], as shown in Fig. 1. All results obtained by three methods are in excellent agreement. In addition, the derived expressions allow a low computational cost compared with the filament method and the finite element method to a specific accuracy. In this paper, the calculated results of self-inductance of Bitter coil and

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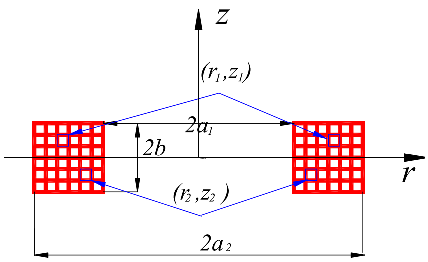

Fig. 1 Circular coil cross-section of Bitter coil

mutual inductance between Bitter coils with three methods are introduced.

2 Self-inductance of Bitter Coil

The self-inductance of a coil can be obtained by the total stored energy expressions. For a Bitter coil, as shown in Fig. 1, by neglecting the effect of turn insulation and the cooling hole, the current density of the two elementary current loops (r_1, z_1) and (r_2, z_2) can be expressed as, respectively,

$$j_1 = \frac{NI}{2b\ln(a_2/a_1)}\frac{1}{r_1} \tag{1}$$

$$j_2 = \frac{NI}{2b\ln(a_2/a_1)}\frac{1}{r_2} \tag{2}$$

The current of elementary loops at (r_1, z_1) and (r_2, z_2) can be expressed by

$$i_1 = j_1\,dr_1\,dz_1 \tag{3}$$

$$i_2 = j_2\,dr_2\,dz_2 \tag{4}$$

Similar to the calculation methods of the total stored magnetic energy with uniform current density proposed by Du et al. [2], the total stored energy of a Bitter coil can be expressed by

$$\varepsilon = \frac{\mu_0 N^2 I^2}{8\ln(a_2/a_1)^2 b^2}\int_{a_1}^{a_2} dr_1 \int_{a_1}^{a_2} dr_2 \int_{-b}^{b} dz_1 \int_{-b}^{b} dz_2 \times \int_0^{\pi} \frac{\cos\phi}{\sqrt{(z_2-z_1)^2 + r_1^2 + r_2^2 - 2r_1 r_2\cos\phi}}\,d\phi \tag{5}$$

where a $(k = 1, 2)$ is the inner and outer radii of the Bitter coil, $2b$ is the length of the coil, μ_0 is the magnetic permeability, N is the number of turns of coil and I is the current of coil.

From the expressions (5) and (6) of the total stored energy, the self-inductance of a Bitter coil can be expressed as (7):

$$\varepsilon = \frac{1}{2}LI^2 \tag{6}$$

$$L = \frac{\mu_0 N^2}{4\ln(a_2/a_1)^2 b^2}\int_{a_1}^{a_2} dr_1 \int_{a_1}^{a_2} dr_2 \int_{-b}^{b} dz_1 \int_{-b}^{b} dz_2 \times \int_0^{\pi} \frac{\cos\phi}{\sqrt{(z_2-z_1)^2 + r_1^2 + r_2^2 - 2r_1 r_2\cos\phi}}\,d\phi \tag{7}$$

After integrating with respect to r_2, z_1, and z_2, the semi-analytical expressions of self-inductance of a Bitter coil with rectangular cross-section can be expressed as

$$L = \frac{\mu_0 N^2}{2b^2[\ln(a_2/a_1)]^2}\sum_{k=1,2}^{2}(-1)^k \int_0^{\pi}\int_{a_1}^{a_2}\left\{\frac{\xi^2}{2}\ln\left(\frac{\varepsilon+\eta}{\varepsilon+\varsigma}\right) + \frac{\varepsilon}{2}(\eta-\varsigma) + 2br\sin\phi\left[\arctan\left(\frac{\varsigma+\varepsilon+2b}{\xi}\right) - \arctan\left(\frac{\varsigma+\varepsilon-2b}{\xi}\right)\right] + b\varepsilon\ln\left(\frac{\eta+2b}{\eta-2b}\right) + 2b^2\ln(\varepsilon+\varsigma)\right\}\cos\phi\,d\phi\,dr \tag{8}$$

where $\varepsilon = r_k - r\cos\phi$, $\xi = r\sin\phi$, $\eta = \sqrt{\varepsilon^2+\xi^2}$, $\varsigma = \sqrt{\varepsilon^2+\xi^2+4b^2}$, a $(k = 1, 2)$ is the inner and outer radii of the Bitter coil, z_k $(k = 1, 2)$ is the lower and upper dimensions of the coil and $2b$ is the length of the coil.

3 Mutual Inductance Between Two Coaxial Bitter Coils

The mutual inductance between two coils can be expressed as the following expression [8]:

$$M = \frac{1}{I_1 I_2}\int_{A_1}\int_{A_2} M_{1,2} J_1 J_2\,dA_1\,dA_2; \tag{9}$$

where A_1 and A_2 are the cross-section areas of two coils, I_1, I_2, J_1, and J_2 are the currents and the current densities of two coils, respectively, and $M_{1,2}$ is their mutual inductance between two filaments. The current densities of two coaxial Bitter coils with rectangular cross-section can be expressed, respectively, as

$$J_1 = \frac{n_1 I_1}{(z_2-z_1)\ln(r_2/r_1)}\frac{1}{r_1'} \tag{10}$$

$$J_2 = \frac{n_2 I_2}{(z_4-z_3)\ln(r_4/r_3)}\frac{1}{r_2'} \tag{11}$$

The mutual inductance between two filaments can be expressed as [9]

$$M_{1,2} = \mu_0 r_1' r_2' \int_0^{\pi} \frac{\cos\theta\,d\theta}{\sqrt{r_1'^2 + r_2'^2 - 2r_1' r_2'\cos\theta + (z_2'-z_1')^2}} \tag{12}$$

where r'_1, r'_2, z'_1, and z'_2 are the radii and heights of two circular coils, respectively.

Substituting (10)–(12) into (9), the mutual inductance between two Bitter coils with rectangular cross section can be expressed as

$$M=\frac{\mu_0 n_1 n_2}{(z_4-z_3)(z_2-z_1)\ln(r_4/r_3)\ln(r_2/r_1)}\int_{r_3}^{r_4}\int_{r_1}^{r_2}\int_{z_3}^{z_4}\int_{z_1}^{z_2}\int_0^{\pi}\frac{\cos\theta}{\sqrt{r_1'^2+r_2'^2-2r'_1r'_2\cos\theta+(z'_2-z'_1)^2}}\,d\theta\,dz'_1\,dz'_2\,dr'_1\,dr'_2 \tag{13}$$

After integrating with respect to z'_1, z'_2, and r'_1, we obtain the following semi-analytical expressions of mutual inductance between coaxial Bitter coils. The expressions can be expressed as

$$M=\alpha\int_{r_3}^{r_4}\int_0^{\pi}\sum_{i=3}^{4}\sum_{j=1}^{2}\sum_{k=1}^{2}(-1)^{i+j+k}\cos\theta(f+g+h)\,d\theta\,dr \tag{14}$$

where

$$f=-\varsigma\gamma\left[\arctan\left(\frac{\varepsilon}{\gamma}\right)-\arctan\left(\frac{\varepsilon\varsigma}{\beta\gamma}\right)\right],\qquad g=\varepsilon\left[\varsigma+\frac{\beta}{2}-\varsigma\ln|\varsigma+\beta|\right],\qquad h=\frac{1}{2}(\gamma^2-\varsigma^2)\ln|\varepsilon+\beta|,$$

$$\alpha=\frac{\mu_0 n_1 n_2}{(z_4-z_3)(z_2-z_1)\ln(r_4/r_3)\ln(r_2/r_1)},$$

$$\beta=\sqrt{\varsigma^2+\varepsilon^2+\gamma^2},\quad \gamma=r\sin\theta,\quad \varepsilon=r_k-r\cos\theta,\quad \varsigma=z_i-z_j,\quad r_k\ (k=1,2)$$

is the inner and outer radii of the Bitter coil of the first coil, r_k $(k=3,4)$ are the inner and outer radii of the Bitter coil of the second coil, z_k $(k=1,2)$ are the lower and upper dimensions of the first coil, and z_k $(k=3,4)$ are the lower and upper dimensions of the second coil.

4 Comparison with Coaxial Cases for Mutual Inductance Calculation

In order to evaluate the accuracy of the semi-analytical expressions for calculating the mutual inductance between Bitter coils, the mutual inductance expressions based on the complete elliptic integral of the first kind and the second kind with filament method between two circular coils and the finite element method with ANSYS software were adopted. The mutual inductance between two circular coils can be expressed as [10]

$$M_{\mathrm{I,II}}=\frac{2\mu_0}{k}\sqrt{r_{\mathrm{I}}r_{\mathrm{II}}}\left[\left(1-\frac{1}{2}k^2\right)K(k)-E(k)\right] \tag{15}$$

where

$$k=\sqrt{\frac{4r_{\mathrm{I}}r_{\mathrm{II}}}{(r_{\mathrm{I}}+r_{\mathrm{II}})^2+d^2}},$$

$K(k)$ and $E(k)$ are the complete elliptic integrals of the first kind and of the second kind, r_{I} and r_{II} are the radii of the first circular coil and the second circular coil, and d is the axial distance between coil planes.

For two coaxial Bitter coils system, we divide the first Bitter coil into $(2p+1)\times(2s+1)$ cells, and the second coil into $(2m+1)\times(2n+1)$ cells, as shown in Fig. 2. Each cell represents one filament and carries the uniform current density. The total mutual inductance between two coaxial Bitter coils with filament method can be expressed by

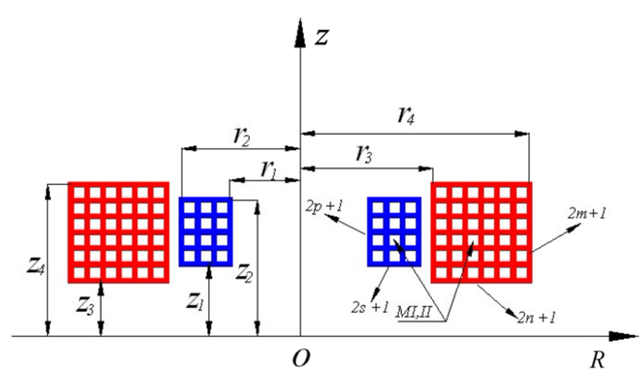

Fig. 2 Circular coil cross-section of two Bitter coils

Table 1 Comparison computational efficiency (filament method)

M_{S-A} (10^{-4} H)	M_{FM} (10^{-4} H)	Computational time (s)	Discrepancy $e_{S-A,F} = \frac{\lvert M_{S-A}-M_F\rvert}{M_{S-A}}$
2.92893120 (time < 0.01 s)	2.93562032	0.03125	2.2838E-03
	2.93163277	0.10938	9.2237E-04
	2.92967248	1.12500	2.5309E-04
	2.92912675	14.96875	6.6765E-05
	2.92898247	227.65625	1.7505E-05
	2.92896473	552.43750	1.1448E-05

$$M = \frac{N_1 N_2 \sum_{g=-p}^{g=p} \sum_{h=-s}^{h=s} \sum_{i=-m}^{i=m} \sum_{j=-n}^{j=n} M_{\text{I,II}}(g,h,i,j)}{(2p+1)(2s+1)(2m+1)(2n+1)} \times \frac{(r_2-r_1)(r_4-r_3)}{\ln(r_4/r_3)\ln(r_2/r_1) r_{\text{I}}(h) r_{\text{II}}(h)} \tag{16}$$

where

$$M_{\text{I,II}}(g,h,i,j) = \frac{2\mu_0}{k(g,h,i,j)} \sqrt{r_{\text{I}}(h) r_{\text{II}}(j)} \left[\left(1 - \frac{1}{2} k^2(g,h,i,j)\right) \times K\big(k(g,h,i,j)\big) - E\big(k(g,h,i,j)\big) \right]$$

$$k(g,h,i,j) = \sqrt{\frac{4 r_{\text{I}}(h) r_{\text{II}}(j)}{[r_{\text{I}}(h) + r_{\text{II}}(j)]^2 + z^2(g,i)}};$$

$$r_{\text{I}}(h) = \frac{r_2+r_1}{2} + \frac{r_2-r_1}{2s+1} h, \quad h = -s, \dots, 0, \dots, s;$$

$$r_{\text{II}}(j) = \frac{r_3+r_4}{2} + \frac{r_4-r_3}{2n+1} j, \quad j = -n, \dots, 0, \dots, n;$$

$$z(g,i) = \frac{1}{2}(z_4+z_3-z_2-z_1) - \frac{z_2-z_1}{(2p+1)} g + \frac{z_4-z_3}{(2m+1)} i$$

$$g = -p, \dots, 0, \dots, p, \quad i = -m, \dots, 0, \dots, m$$

To verify the validity of the semi-analytic expressions, the mutual inductance between two Bitter coils was calculated. The dimensions of the two Bitter coils as a test example are as follows: $r_1 = 0.025$ m, $r_2 = 0.035$ m, $r_3 = 0.045$ m, $r_4 = 0.065$ m, $z_2 - z_1 = 0.040$ m, $z_4 - z_3 = 0.060$ m, $n_1 = n_2 = 100$, and $c = 0$ is the axial distance between two thick coil mid-planes.

Table 1 lists the calculated results by the two methods. The semi-analytical expression gives the mutual inductance of $2.92893120 \times 10^{-4}$ H. The finite element method with ANSYS software was used to calculate the mutual inductance between two Bitter coils. The finite element method gives the mutual inductance of 2.9024×10^{-4} H with 29.09 s. All results computed by three methods are in excellent agreement.

Table 2 Comparison computational efficiency (filament method)

L_{FM} (10^{-4} H)	Cells	Computational time (s)	Discrepancy $e_{S-A,F} = \frac{\lvert M_{S-A}-M_F\rvert}{M_{S-A}}$
4.280545	20 × 20	0.0402	2.4164E-02
4.352049	40 × 40	0.1406	7.3365E-03
4.374432	80 × 80	0.7969	2.1822E-03
4.381201	160 × 160	11.4375	6.3384E-04
4.383192	320 × 320	182.0000	1.7932E-04

5 Self-inductance Calculation with Filament Method

In order to evaluate the accuracy of the obtained self-inductance expression for a Bitter coil with rectangular cross-section, the filament method and the finite element method are considered for comparing the above semi-analytical expression of self-inductance. As given in (6), the self-inductance of a Bitter coil can be obtained from the total stored energy. The total stored energy consists of the self-energy of each filament and the mutual energy between filaments, as described in (17):

$$\varepsilon_m = \frac{1}{2} \sum_{k=1}^{n} L_k I_k^2 + \frac{1}{2} \sum_{\substack{h,k=1 \\ (h \neq k)}}^{n} M_{h,k} I_h I_k \tag{17}$$

For a circular filament, the self-inductance can be given as

$$L = \mu_0 r \left(\ln \frac{8r}{a} - 1.75 \right) \tag{18}$$

For two coaxial filaments, the mutual inductance can be obtained by (15).

To evaluate the accuracy of the semi-analytical expressions of the self-inductance, the self-inductance of Bitter coil with three methods was calculated. The dimensions of the Bitter coil as a test example are as follows: $r_1 = 0.025$ m, $r_2 = 0.035$ m; $z_2 - z_1 = 0.04$ m; $N = 100$. Table 2 lists the calculated results by the two methods. The semi-analytical expressions and the finite element method with ANSYS give the self-inductance of 4.383978×10^{-4} H with 1.56×10^{-2} s and 4.4528×10^{-4} H with 27.02 s, respectively. The semi-analytical expressions give a low computational cost compared to the filament method and the finite element method to a given accuracy.

6 Conclusion

The inductance of Bitter coil with rectangular cross-section based on semi-analytical expressions has been calculated.

The inductance has been implemented using Gauss numerical integration method with FORTRAN programming. In order to evaluate the accuracy of both semi-analytical expressions, the inductance has also been calculated with filament method and finite element method. Results obtained by three methods are in excellent agreement. The calculation results have shown that both derived semi-analytical expressions allow a low computational cost comparing with the filament method and the finite element method to a specific accuracy. The derived mutual inductance expressions can also be used to calculate the axial force between coaxial Bitter coils with mutual inductance gradient method.

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