Nonlinear canonical gyrokinetic Vlasov equation and computation of the gyrocenter motion in tokamaks

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The nonlinear canonical gyrokinetic Vlasov equation is obtained from the nonlinear noncanonical gyrokinetic theory using the property of the coordinate transform. In the linear approximation, it exactly recovers the previous linear canonical gyrokinetic equations derived by the Lie-transform perturbation method. The computation of the test particle gyrocenter motion in tokamaks with a large magnetic perturbation is presented and discussed. The numerical results indicate that the second-order gyrocenter Hamiltonian is important for the gyrocenter motion of the trapped electron in tokamaks with a large magnetic perturbation. © *2013 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4789550]

I. INTRODUCTION

Gyrokinetic theory[1–12] and simulation[13–20] are widely used to investigate the long time behavior of the magnetized plasma, including the low-frequency turbulence and the interaction between the high-frequency wave and particles in tokamaks. The classical gyrokinetic theory[1–3] has been developed by the recursive method, while the modern gyrokinetic theory[4–11] has been developed by the Lie-transform perturbation method.[21–24]

The Lie-transform perturbation method is used to simplify the equations of motion. It has been applied to the guiding-center (GC) theory[25–27] and the modern gyrokinetic theory. In the modern gyrokinetic theory, the phase-space transformation from the GC coordinates to the gyrocenter (GY) (GC in perturbed fields) coordinates is a Lie transform, which is dependent of the electromagnetic perturbations. The transformation decouples gyromotion from the GY motion, that is, the GY magnetic moment is a conserved quantity.

The modern gyrokinetic theory based on the Lie-transform perturbation method has been derived by starting from different GC coordinates; for example, the noncanonical coordinates[4–8] and the canonical coordinates[9–11] have been used independently to derive the gyrokinetic theory. The noncanonical coordinates are simple and clear, while the canonical coordinates are useful in some circumstances, such as the numerical computation of the equations of motion[11,28] and the canonical equilibrium distribution function.[29,30] It has been found that the zonal flow damping can be correctly simulated using the canonical equilibrium distribution function, while spurious zonal flow oscillations can be generated by the local (noncanonical) equilibrium distribution function.[29,30] The previous canonical gyrokinetic theories[9–11] are linear and written in terms of specific canonical coordinates. It is of interest to develop the nonlinear gyrokinetic theory in terms of canonical coordinates.

In this paper, the nonlinear canonical gyrokinetic Vlasov equation using the property of the coordinate transform and its numerical application is presented. The rest of the paper is organized as follows. In Sec. II, the nonlinear gyrokinetic theory based on the Lie-transform perturbation method is reviewed. In Sec. III, the property of the Lie transform in the nonlinear gyrokinetic theory is presented. In Sec. IV, the canonical nonlinear gyrokinetic equations are presented using the property of the coordinate transform. In Sec. V, the computation of the test particle gyrocenter motion with a large magnetic perturbation in tokamaks is presented and discussed. In Sec. VI, the main results are summarized.

II. A BRIEF REVIEW OF NONLINEAR GYROKINETIC THEORY BASED ON THE LIE-TRANSFORM PERTURBATION METHOD

In the gyrokinetic theory, the following standard orderings are assumed,[8]

$$\epsilon_B = \rho_0/L \ll 1, \tag{1a}$$

$$\frac{|\delta \boldsymbol{E}|}{v_{th}B_0} \sim \frac{|\delta \boldsymbol{B}|}{B_0} \sim \frac{\delta f}{f_0} \sim \epsilon_\delta \ll 1. \tag{1b}$$

Here, ρ_0 is the Larmor radius, and L is the characteristic length of the equilibrium magnetic field B_0; $(\delta\boldsymbol{E}, \delta\boldsymbol{B})$ are the perturbation parts of the electromagnetic fields defined as $(\delta\boldsymbol{B} = \nabla\times\delta\boldsymbol{A},\ \delta\boldsymbol{E} = -\nabla\delta\phi - \partial_t\delta\boldsymbol{A})$, and v_{th} is the thermal velocity of the particle. δf and f_0 are the perturbation part and the unperturbed part of the distribution function, respectively.

A. Unperturbed guiding-center equations of motion

The unperturbed fundamental one-form (the GC extended Lagrangian) can be written in terms of the noncanonical GC coordinates $(\boldsymbol{X}, v_\parallel, \xi, \mu, t, -U)$ as

$$\hat{\Gamma}_0 \equiv \Gamma_0 - h_0 d\tau = \Gamma_{0i}dZ^i - h_0 d\tau \tag{2a}$$

$$= (v_\parallel \boldsymbol{b}_0 + \boldsymbol{A}_0)\cdot d\boldsymbol{X} + \mu d\xi - U\,dt - h_0 d\tau, \tag{2b}$$

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with $\boldsymbol{X}$ the GC position, $v_\parallel$ the parallel velocity, μ the magnetic moment, ξ the gyro-angle, U the total energy of the particle, and τ the independent parameter. Here, $\boldsymbol{b}_0 = \boldsymbol{B}_0/B_0$, with $\boldsymbol{B}_0$ the equilibrium magnetic field and $\boldsymbol{X} = \boldsymbol{r} - \boldsymbol{\rho}_0(\mu, \xi)$, with $\boldsymbol{r}$ the particle position and $\boldsymbol{\rho}_0$ the Larmor radius vector. Throughout the paper, $e_s = 1 = m_s$ is set to simplify the formulae, with e_s and m_s the electric charge and the mass of the particle species s, respectively. The unperturbed GC extended Hamiltonian is

$$h_0 \equiv H_0 - U = \frac{1}{2}v_\parallel^2 + \mu B_0 + \phi_0 - U, \tag{3}$$

with H_0 the unperturbed GC Hamiltonian, and ϕ_0 the equilibrium scalar potential. The Lagrange two-form is defined as $\omega \equiv d\Gamma$.[31] The unperturbed Lagrange two-form can be written as

$$\hat{\omega}_0 \equiv \omega_0 - dh_0 \wedge d\tau \tag{4a}$$

$$= \frac{1}{2}\epsilon_{ijk}B_0^{*k} dX^i \wedge dX^j + b_{0j} dv_\parallel \wedge dX^j + d\mu \wedge d\xi + d(-U)\wedge dt - dh_0 \wedge d\tau, \tag{4b}$$

with

$$\boldsymbol{B}_0^* = \boldsymbol{B}_0 + v_\parallel \nabla \times \boldsymbol{b}_0, \tag{5}$$

d the exterior differential, $\wedge$ the exterior product, ϵ_{ijk} the permutation tensor, and b_{0j} the component of the unit vector $\boldsymbol{b}_0$. The Jacobian of the noncanonical GC coordinates $\mathcal{J}_0$ can be obtained from $\mathcal{J}_0^2 = |\omega_{0ij}|$, that is, $\mathcal{J}_0 = B_{\parallel 0}^* = \boldsymbol{B}_0^* \cdot \boldsymbol{b}_0$. The Jacobian of the canonical GC coordinates is unity.

The unperturbed GC equations of motion in terms of the noncanonical coordinates are written as

$$\frac{d_0 Z^i}{d\tau} = \{Z^i, h_0\}_0 = J_0^{ij}\partial_j h_0. \tag{6}$$

Here, $\boldsymbol{J}_0$ is the unperturbed Poisson matrix, which is the inverse matrix of the unperturbed Lagrange matrix $\boldsymbol{\omega}_0$. The non-zero components of the unperturbed Poisson matrix are

$$J_0^{X^i X^j} = -\frac{\epsilon^{ijk} b_{0k}}{B_{\parallel 0}^*}, \tag{7a}$$

$$J_0^{X^i v_\parallel} = -J_0^{v_\parallel X^i} = \frac{B_0^{*i}}{B_{\parallel 0}^*}, \tag{7b}$$

$$J_0^{\xi\mu} = -J_0^{\mu\xi} = 1, \tag{7c}$$

$$J_0^{t(-U)} = -J_0^{-Ut} = 1. \tag{7d}$$

B. Lie-transform perturbation method

When a perturbation of the electromagnetic fields is introduced, the GC magnetic moment μ is not conservative any more. To decouple the GY motion from the gyromotion, we need to find a new conservative magnetic moment $\bar{\mu}$. The phase-space Lagrangian Lie-transform perturbation method[22–24] is an effective method for seeking a new conservative magnetic moment by the GY phase-space transformation. The GY phase-space transformation $\mathcal{T}_{gy}$ and its inverse $\mathcal{T}_{gy}^{-1}$ are defined as

$$\bar{Z}^i(Z; \epsilon_\delta) \equiv \mathcal{T}_{gy} Z^i, \tag{8a}$$

$$Z^i(\bar{Z}; \epsilon_\delta) \equiv \mathcal{T}_{gy}^{-1} \bar{Z}^i. \tag{8b}$$

$\mathcal{T}_{gy}$ and $\mathcal{T}_{gy}^{-1}$ are generated by the nth-order vector fields $\boldsymbol{G}_n$ $(n = 1, 2\cdots)$. Due to the scalar invariance,

$$\bar{\mathcal{F}}(\bar{Z}) = \mathcal{F}(Z), \tag{9}$$

a push-forward operator T_{gy}^{-1} and a pull-back operator T_{gy} induced by the transformation and its inverse are written as

$$\bar{\mathcal{F}} = T_{gy}^{-1}\mathcal{F}, \tag{10a}$$

$$\mathcal{F} = T_{gy}\bar{\mathcal{F}}. \tag{10b}$$

T_{gy} and T_{gy}^{-1} are expressed by the generating vectors $\boldsymbol{G}_n$. The transformation of the fundamental one-form is

$$\bar{\hat{\Gamma}} = T_{gy}^{-1}\hat{\Gamma} + dS, \tag{11}$$

where S is a gauge function used to simplify the equations of motion. The scalar function $\mathcal{F}$ can be chosen as the Hamiltonian and the distribution function. From Eqs. (8), (9), and (10), we have

$$T_{gy}\bar{\mathcal{F}}(Z) = \bar{\mathcal{F}}(\mathcal{T}_{gy} Z), \tag{12a}$$

$$T_{gy}^{-1}\mathcal{F}(\bar{Z}) = \mathcal{F}(\mathcal{T}_{gy}^{-1}\bar{Z}). \tag{12b}$$

Note that $\{F, h\}_Z = \{\bar{F}, \bar{h}\}_{\bar{Z}}$ and $\{Z^i, h\}_Z = \frac{\partial Z^i}{\partial \bar{Z}^j}(\bar{Z})\{\bar{Z}^j, \bar{h}\}_{\bar{Z}}$.

For simplying the GY equations of motion, the symplectic part of the transformed one-form has the same function as the symplectic part of the unperturbed one, therefore,

$$\bar{\boldsymbol{\omega}}(\bar{Z}) = \boldsymbol{\omega}_0(Z)|_{Z=\bar{Z}}. \tag{13}$$

Note that $\boldsymbol{J} = \boldsymbol{\omega}^{-1}$ and $\mathcal{J}^2 = |\omega_{ij}|$,[26] one can easily find

$$\bar{\boldsymbol{J}}(\bar{Z}) = \boldsymbol{J}_0(Z)|_{Z=\bar{Z}}, \tag{14a}$$

$$\bar{\mathcal{J}}(\bar{Z}) = \mathcal{J}_0(Z)|_{Z=\bar{Z}}. \tag{14b}$$

Note that the GC coordinates and the GY coordinates can be chosen as the canonical coordinates or the noncanonical coordinates. The Jacobian of the GY coordinates $\bar{\mathcal{J}}$ has the same function as the unperturbed Jacobian of the GC coordinates $\mathcal{J}_0$,

$$\bar{\mathcal{J}}(\bar{Z}) = \mathcal{J}_0(Z)|_{Z=\bar{Z}} = B_{\parallel 0}^*(Z)|_{Z=\bar{Z}}, \tag{15}$$

for the noncanonical coordinates $(\boldsymbol{X}, v_\parallel, \xi, \mu, t, -U)$;

$$\bar{\mathcal{J}}(\bar{Z}) = \mathcal{J}_0(Z)|_{Z=\bar{Z}} = 1, \tag{16}$$

for the canonical coordinates.[9–11]

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C. Noncanonical gyrokinetic equations based on the Lie-transform perturbation method

With the perturbed scalar and vector potentials $(\delta\phi, \delta\boldsymbol{A})$ introduced, the fundamental one-form can be separated into the unperturbed part and the perturbed part,

$$\hat{\Gamma} \equiv \hat{\Gamma}_0 + \hat{\Gamma}_1, \tag{17}$$

where the perturbed part of the one-form is written as

$$\begin{aligned}\hat{\Gamma}_1 = \Gamma_1 &= \Gamma_{1i} dZ^i \\ &= \delta\boldsymbol{A}(\boldsymbol{X}+\boldsymbol{\rho}_0, t)\cdot d(\boldsymbol{X}+\boldsymbol{\rho}_0) - \delta\phi(\boldsymbol{X}+\boldsymbol{\rho}_0, t)dt.\end{aligned} \tag{18}$$

Here, the spatial dependence of the gyroradius vector $\boldsymbol{\rho}_0$ is ignored, that is, $\boldsymbol{\rho}_0 = \boldsymbol{\rho}_0(\mu, \xi)$.[8]

The GY extended-phase-space transformation can be expanded in powers of the amplitude ordering parameter ϵ_δ up to $O(\epsilon_\delta^2)$, written as

$$\bar{Z}^i = Z^i + G_1^i + G_2^i + \frac{1}{2}G_1^j \partial_j G_1^i. \tag{19}$$

Note that t is not affected by the transformation, that is, $G_1^t = 0 = G_2^t$. The transformed fundamental one-form is defined as

$$\hat{\bar{\Gamma}} \equiv \bar{\Gamma} - \bar{h} d\tau = \bar{\Gamma}_i d\bar{Z}^i - \bar{h} d\tau. \tag{20}$$

It can be expanded in powers of ϵ_δ up to $O(\epsilon_\delta^2)$, written as

$$\bar{\Gamma}_{0i} = \Gamma_{0i}, \tag{21a}$$

$$\bar{\Gamma}_{1i} = \Gamma_{1i} - G_1^j \omega_{0ji} + \partial_i S_1, \tag{21b}$$

$$\bar{\Gamma}_{2i} = -G_2^j \omega_{0ji} - \frac{1}{2}G_1^j(\omega_{1ji} + \bar{\omega}_{1ji}) + \partial_i S_2, \tag{21c}$$

$$\bar{h}_0 = h_0, \tag{21d}$$

$$\bar{h}_1 = -G_1^i \partial_i h_0, \tag{21e}$$

$$\bar{h}_2 = -G_2^i \partial_i \bar{h}_0 - \frac{1}{2}G_1^i \partial_i \bar{h}_1. \tag{21f}$$

Here, S_n is independent of U and τ, that is, $\partial_U S_n = 0 = \partial_\tau S_n$. ω_1 and $\bar{\omega}_1$ are Lagrange two-forms, defined as

$$\omega_{1ji} = \partial_j \Gamma_{1i} - \partial_i \Gamma_{1j}, \tag{22a}$$

$$\bar{\omega}_{1ji} = \partial_j \bar{\Gamma}_{1i} - \partial_i \bar{\Gamma}_{1j}. \tag{22b}$$

Here, S_1 and S_2 are the first-order and the second-order of the scalar field function used for canceling the gyroangle dependence of the extended Lagrangian. The first-order Lagrange two-form can be rewritten as

$$\begin{aligned}\omega_1 = d\Gamma_1 &= \frac{1}{2}\epsilon_{ijk}\delta B^k dX^i \wedge dX^j + \delta E_j dX^j \wedge dt \\ &+ \epsilon_{ijk}\delta B^k \partial_\mu \rho_0^i d\mu \wedge dX^j + \epsilon_{ijk}\delta B^k \partial_\xi \rho_0^i d\xi \wedge dX^j \\ &+ \delta\boldsymbol{E}\cdot\partial_\mu \boldsymbol{\rho}_0 d\mu \wedge dt + \delta\boldsymbol{E}\cdot\partial_\xi \boldsymbol{\rho}_0 d\xi \wedge dt.\end{aligned} \tag{23}$$

Clearly, ω_1 denotes the perturbation part of the electromagnetic field tensor. From Eqs. (21b) and (21c), the first-order and second-order generating vector fields are obtained,

$$G_1^i = [\partial_j S_1 + (\Gamma_{1j} - \bar{\Gamma}_{1j})]J_0^{ji}, \tag{24a}$$

$$G_2^i = \left[\partial_j S_2 - \frac{1}{2}G_1^k(\omega_{1kj} + \bar{\omega}_{1kj})\right]J_0^{ji}. \tag{24b}$$

For simplifying the GY equations of motion, the simplectic part of the transformed Lagrangian is chosen to be formally same as the unperturbed one, that is, $\bar{\Gamma}_{1i} = 0 = \bar{\Gamma}_{2i}$, then $\bar{\omega}_1 = 0$, $\bar{\omega} = \omega_0$. The nth-order generating vectors $\boldsymbol{G}_n$ in terms of the GC coordinates can be obtained from Eq. (24)

$$G_n^i = \{S_n, Z^i\} + \delta\mathcal{A}_n \cdot \{\boldsymbol{X}+\boldsymbol{\rho}_0, Z^i\} - \delta\psi_n\{t, Z^i\}, \tag{25}$$

where

$$[\delta\mathcal{A}_1, \delta\psi_1] = [\delta\boldsymbol{A}, \delta\phi], \tag{26a}$$

$$[\delta\mathcal{A}_2, \delta\psi_2] = \left[\frac{1}{2}G_1^{\boldsymbol{r}} \times \delta\boldsymbol{B}, \frac{1}{2}G_1^{\boldsymbol{r}}\cdot\delta\boldsymbol{E}\right], \tag{26b}$$

$$G_1^{\boldsymbol{r}} = \{S_1, \boldsymbol{X}+\boldsymbol{\rho}_0\}. \tag{26c}$$

Note that the generating vectors $\boldsymbol{G}_n$ are non-Hamiltonian flows, although the first term in Eq. (25) is a Hamiltonian flow. However, $\boldsymbol{G}_n$ are incompressible flows in the phase space.[12]

Using Eq. (25), the nth-order generating vectors $\boldsymbol{G}_n$ can be rewritten as

$$G_n^{\boldsymbol{X}} = -\frac{\boldsymbol{b}_0}{B_{\parallel 0}^*}\times(\delta\mathcal{A}_n + \nabla S_n) - \partial_{v_\parallel} S_n \frac{\boldsymbol{B}_0^*}{B_{\parallel 0}^*}, \tag{27a}$$

$$G_n^{v_\parallel} = \frac{\boldsymbol{B}_0^*}{B_{\parallel 0}^*}\cdot(\delta\mathcal{A}_n + \nabla S_n), \tag{27b}$$

$$G_n^{\xi} = -(\delta\mathcal{A}_n \cdot \partial_\mu \boldsymbol{\rho}_0 + \partial_\mu S_n), \tag{27c}$$

$$G_n^{\mu} = \delta\mathcal{A}_n \cdot \partial_\xi \boldsymbol{\rho}_0 + \partial_\xi S_n, \tag{27d}$$

$$G_n^{U} = \delta\psi_n - \partial_t S_n, \tag{27e}$$

$$G_n^{t} = 0. \tag{27f}$$

To decouple the GY motion from the gyromotion, the GY Hamiltonian $\bar{h}_n$ is chosen to satisfy the condition $\bar{h}_n = \langle\bar{h}_n\rangle$. Here, $\langle\cdots\rangle$ denotes the gyro-average. Thus, the first-order and second-order scalar field functions can be chosen as

$$\frac{d_0 S_1}{d\tau} = \tilde{K}_1, \tag{28a}$$

$$\begin{aligned}\frac{d_0\left(S_2 + \frac{1}{2}\{S_1, Z^i\}\Gamma_{1i}\right)}{d\tau} = &-\frac{1}{2}\{S_1, \widetilde{(\dot{S}_1)_0}\} + \frac{e_s^2}{2m_s}\widetilde{|\delta\boldsymbol{A}|^2} \\ &- \{S_1, \bar{h}_1\},\end{aligned} \tag{28b}$$

with K_n defined as

$$K_1 \equiv -\Gamma_{1i}\dot{Z}_0^i = \delta\phi - (\dot{\boldsymbol{X}}_0 + \dot{\xi}_0 \partial_\xi \boldsymbol{\rho}_0)\cdot\delta\boldsymbol{A}, \tag{29a}$$

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$$K_2 \equiv \frac{1}{2} G_1^i \omega_{1ij} \dot{Z}_0^j. \tag{29b}$$

For any function A, $\tilde{A}$ denotes the gyroangle dependence part, defined as $\tilde{A} \equiv A - \langle A \rangle$.

By substituting Eq. (28) into Eqs. (21e) and (21f), the first-order and the second-order GY extended Hamiltonians are written as

$$\bar{h}_1 = \langle K_1 \rangle = \langle \delta\phi \rangle - \dot{\boldsymbol{X}}_0 \cdot \langle \delta \boldsymbol{A} \rangle - \dot{\xi}_0 \langle \delta \boldsymbol{A} \cdot \partial_\xi \boldsymbol{\rho}_0 \rangle, \tag{30a}$$

$$\bar{h}_2 = -\frac{1}{2} \langle \{ S_1, (\dot{S}_1)_0 \} \rangle + \frac{1}{2} \langle |\delta \boldsymbol{A}|^2 \rangle. \tag{30b}$$

The GY equations of motion are written in terms of the Poisson matrix as

$$\frac{d\bar{Z}^i}{d\tau} = J_0^{ij} \partial_j \bar{h} = J_0^{ij} \partial_j (\bar{h}_0 + \bar{h}_1 + \bar{h}_2). \tag{31}$$

Note that the GY equations of motion satisfy the Liouville's theorem.[8]

The nonlinear gyrokinetic Vlasov equation $\{\bar{F}, \bar{h}\} = 0$ can be written in terms of the noncanonical coordinates as

$$\partial_t \bar{F} + \frac{d\bar{\boldsymbol{X}}}{dt} \cdot \bar{\nabla} \bar{F} + \frac{d\bar{v}_\parallel}{dt} \partial_{\bar{v}_\parallel} \bar{F} = 0. \tag{32}$$

The GC distribution function can be obtained by the pull-back transformation of the GY distribution function, that is, $F = T_{gy} \bar{F}$,

$$F = \bar{F} + G_1^i \partial_i \bar{F} + G_2^i \partial_i \bar{F} + \frac{1}{2} G_1^i \partial_i (G_1^j \partial_j \bar{F}). \tag{33}$$

Note that in gyrokinetic theories[8] and simulations,[20] Eq. (33) is usually kept up to $O(\epsilon_\delta)$.

For the low-frequency turbulence, the displacement current can be neglected, thus the gyrokinetic Maxwell equations can be written as

$$\nabla^2 (\phi_0 + \delta\phi) + \partial_t (\nabla \cdot (\boldsymbol{A}_0 + \delta \boldsymbol{A})) = -\frac{1}{\epsilon_0} \sum \rho, \tag{34a}$$

$$\nabla^2 (\boldsymbol{A}_0 + \delta \boldsymbol{A}) - \nabla (\nabla \cdot (\boldsymbol{A}_0 + \delta \boldsymbol{A})) = -\mu_0 \sum \boldsymbol{J}, \tag{34b}$$

where the summation is taken over particle species and the particle charge density and current are expressed in terms of the guiding-center distribution

$$\rho = \int F \delta (\boldsymbol{X} + \boldsymbol{\rho}_0 - \boldsymbol{r}) d^6 \boldsymbol{Z}, \tag{35a}$$

$$\boldsymbol{J} = \int (\dot{\boldsymbol{X}}_0 + \dot{\xi}_0 \partial_\xi \boldsymbol{\rho}_0) F \delta (\boldsymbol{X} + \boldsymbol{\rho}_0 - \boldsymbol{r}) d^6 \boldsymbol{Z}, \tag{35b}$$

with $d^6 \boldsymbol{Z} = \frac{B^*_{\parallel 0}}{m} d^3 \boldsymbol{X} dv_\parallel d\mu d\xi$. The energy invariant is defined as

$$E_{total} = \int d^3 x \left(\frac{\epsilon_0}{2} |\boldsymbol{E}_0 + \delta \boldsymbol{E}|^2 + \frac{1}{2\mu_0} |\boldsymbol{B}_0 + \delta \boldsymbol{B}|^2 \right) + \sum \int d^6 Z F \left(\frac{1}{2} v_\parallel^2 + \mu B_0 \right). \tag{36}$$

III. PROPERTY OF THE COORDINATE TRANSFORM

In this section, we present the property of the coordinate transform in the gyrokinetic theory. It will be used in the next section for deriving the nonlinear canonical gyrokinetic equations from the noncanonical gyrokinetic equations.

Let Z_1 and Z_2 be two different sets of GC coordinates, and the transformation between them is given by

$$Z_2 = \mathcal{Z}_2(Z_1). \tag{37}$$

Let

$$\bar{Z}_1 = \mathcal{T}_{gy} Z_1, \tag{38a}$$

$$\bar{Z}_2 = \mathcal{T}_{gy} Z_2, \tag{38b}$$

which means that $\bar{Z}_1$ and $\bar{Z}_2$ are two sets of GY coordinates based on the two sets of GC coordinates Z_1 and Z_2, respectively. $\mathcal{T}_{gy}$ is determined by the generating vectors (25). The transform between the two sets of GY coordinates is given by

$$\bar{Z}_2 = \bar{\mathcal{Z}}_2(\bar{Z}_1). \tag{39}$$

Using the scalar invariance

$$\bar{\mathcal{F}}(\bar{Z}_2) = \mathcal{F}(Z_2), \tag{40}$$

and the pull-back transform, we found

$$\begin{aligned} \bar{\mathcal{F}}(\bar{\mathcal{Z}}_2(\bar{Z}_1)) &= \mathcal{F}(\mathcal{Z}_2(Z_1)) \\ &= T_{gy} \bar{\mathcal{F}}(\mathcal{Z}_2(Z_1)) = \bar{\mathcal{F}}(T_{gy} \mathcal{Z}_2(Z_1)) \\ &= \bar{\mathcal{F}}(\mathcal{Z}_2(\mathcal{T}_{gy} Z_1)) = \bar{\mathcal{F}}(\mathcal{Z}_2(\bar{Z}_1)). \end{aligned} \tag{41}$$

Therefore, we found the property of the coordinate transform

$$\bar{\mathcal{Z}}_2(\bar{Z}_1) = \mathcal{Z}_2(Z_1)|_{Z_1 = \bar{Z}_1}, \tag{42}$$

which indicates that the transform between two sets of the GY coordinates is the same as the transform between the two corresponding GC coordinates.

Using this property, we can quickly derive the GY equations of motion based on an arbitrary GC coordinates (Z_2) from a standard GY equations of motion based on Z_1, instead of carrying out the lengthy calculation of the Lie transform once again from the GC coordinates (Z_2) to the GY coordinates ($\bar{Z}_2$), as will be shown in the following. The point is that the Lie transform from the GC coordinates to the GY coordinates is a time-dependent transform determined by the field perturbations, while the coordinate transform between the GC (or between the GY) coordinates is a time-independent transform that is independent of the field perturbations.

For any vector $\boldsymbol{A}$ in terms of the GC coordinates, we have

$$A^i(Z_2) = A^j(Z_1) \frac{\partial Z_2^i}{\partial Z_1^j}. \tag{43}$$

A^i can be chosen as $\frac{dZ^i}{dt}(Z)$ and $G_n^i(Z)$.

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For any vector $\boldsymbol{A}$ in terms of the GY coordinates, we have

$$\bar{A}^i(\bar{Z}_2) = \bar{A}^j(\bar{Z}_1)\frac{\partial \bar{Z}_2^i}{\partial \bar{Z}_1^j}. \tag{44}$$

Using the property of the coordinate transform (42), we have

$$\frac{\partial \bar{Z}_2^i}{\partial \bar{Z}_1^j} = \left[\frac{\partial Z_2^i}{\partial Z_1^j}\right]_{Z_1=\bar{Z}_1}. \tag{45}$$

By choosing $\bar{A}^i$ as $\frac{d\bar{Z}^i}{dt}(\bar{Z})$ and using Eq. (45), Eq. (44) becomes

$$\frac{d\bar{Z}_2^i}{dt}(\bar{Z}_2) = \frac{d\bar{Z}_1^j}{dt}(\bar{Z}_1)\left[\frac{\partial Z_2^i}{\partial Z_1^j}\right]_{Z_1=\bar{Z}_1} \tag{46a}$$

$$= \bar{J}^{jk}(\bar{Z}_1)\frac{\partial \bar{h}(\bar{Z}_1)}{\partial \bar{Z}_1^k}\left[\frac{\partial Z_2^i}{\partial Z_1^j}\right]_{Z_1=\bar{Z}_1}$$

$$= \left[\frac{\partial Z_2^i}{\partial Z_1^j}\right]_{Z_1=\bar{Z}_1}\bar{J}^{jk}(\bar{Z}_1)\frac{\partial \bar{Z}_2^l}{\partial \bar{Z}_1^k}\frac{\partial \bar{h}'(\bar{Z}_2)}{\partial \bar{Z}_2^l}, \tag{46b}$$

$$= \bar{J}'^{il}(\bar{Z}_2)\frac{\partial \bar{h}'(\bar{Z}_2)}{\partial \bar{Z}_2^l}, \tag{46c}$$

with the GY Poisson matrix and the GY Hamiltonian in terms of the GY coordinates $\bar{Z}_2$ given by

$$\bar{J}'^{il}(\bar{Z}_2) \equiv \left[\frac{\partial Z_2^i}{\partial Z_1^j}J_0^{jk}(Z_1)\frac{\partial Z_2^l}{\partial Z_1^k}\right]_{Z_1=\bar{Z}_1}, \tag{47a}$$

$$= [J_0'^{jk}(Z_2)]_{Z_2=\bar{Z}_2}, \tag{47b}$$

$$\bar{h}(\bar{Z}_1) = \bar{h}(\bar{\mathcal{Z}}_1(\bar{Z}_2)) = [\bar{h}(\mathcal{Z}_1(Z_2))]_{Z_2=\bar{Z}_2} \equiv \bar{h}'(\bar{Z}_2). \tag{47c}$$

Note that in obtaining Eq. (47a), we have used Eq. (14a) to find $\bar{J}^{jk}(\bar{Z}_1) = [J_0^{jk}(Z_1)]_{Z_1=\bar{Z}_1}$.

Equation (47a) indicates that the Poisson matrix in terms of the GY coordinates $\bar{Z}_2$ can be obtained from the Poisson matrix in terms of $\bar{Z}_1$, by simply using the corresponding GC coordinate transform relation. Equation (47b) indicates that the Poisson matrix in terms of the GY coordinates $\bar{Z}_2$ can be obtained by simply making the substitution, $Z_2 \to \bar{Z}_2$, in $\boldsymbol{J}_0(Z_2)$, the unperturbed GC Poisson matrix in terms of Z_2, as is expected using Eq. (14a). Equation (47c) indicates that $\bar{h}'(\bar{Z}_2)$, the GY Hamiltonian in terms of $\bar{Z}_2$, can be obtained from $\bar{h}(\bar{Z}_1)$ by simply using the transform between the GC coordinates, as is expected from the scalar invariance of the GY Hamiltonian.

By choosing $\bar{A}^i$ as $G_n^i(\bar{Z})$, Eq. (44) becomes

$$G_n^i(\bar{Z}_2) = G_n^j(\bar{Z}_1)\left[\frac{\partial Z_2^i}{\partial Z_1^j}\right]_{Z_1=\bar{Z}_1}. \tag{48}$$

Equation (48) means that the generating vectors in terms of the GY coordinates $\bar{Z}_2$ can be obtained from the generating vectors in terms of the GY coordinates $\bar{Z}_1$, using the coordinate transform between the GC coordinates Z_1 and Z_2.

IV. NONLINEAR GYROKINETIC EQUATIONS IN TERMS OF THE CANONICAL COORDINATES

If the gyrokinetic equations in terms of a standard coordinate system are known, one can find the gyrokinetic equations in terms of a new coordinate system by simply using the property of the coordinate transform (42), instead of carrying out the lengthy calculation of the Lie transform from the new GC coordinates to the new GY coordinates.

A. The nonlinear canonical gyrokinetic equations

In Sec. II, the nonlinear gyrokinetic equations in terms of the noncanonical GC coordinates $Z = (\boldsymbol{X}, v_\parallel, \xi, \mu, t, -U)$ using the Lie transform perturbation method have been introduced. In this subsection, we develop the nonlinear gyrokinetic equations based on general canonical GC coordinates $Z^c = (\theta^1, P^1, \theta^2, P^2, \xi, \mu, t, -U)$ by the coordinate transform between $(\boldsymbol{X}, v_\parallel)$ and $(\theta^1, P^1, \theta^2, P^2)$.

The extended-phase-space transformation between the canonical GC coordinates Z^c and their corresponding canonical GY coordinates $\bar{Z}^c = (\bar{\theta}^1, \bar{P}^1, \bar{\theta}^2, \bar{P}^2, \bar{\xi}, \bar{\mu}, t, -\bar{U})$ is $\bar{Z}^c = \mathcal{T}_{gy}Z^c$. It can be expanded in powers of ϵ_δ up to $O(\epsilon_\delta^2)$ and has the same form of Eq. (19). From Eqs. (25) and (48), we can obtain the first-order generating vectors related to $(\theta^1, P^1, \theta^2, P^2)$ in terms of the canonical coordinates

$$G_1^{\theta^1} = -\left(\delta\boldsymbol{A}\cdot\frac{\partial \boldsymbol{X}}{\partial P^1} + \partial_{P^1}S_1^c\right), \tag{49a}$$

$$G_1^{\theta^2} = -\left(\delta\boldsymbol{A}\cdot\frac{\partial \boldsymbol{X}}{\partial P^2} + \partial_{P^2}S_1^c\right), \tag{49b}$$

$$G_1^{P^1} = \delta\boldsymbol{A}\cdot\frac{\partial \boldsymbol{X}}{\partial \theta^1} + \partial_{\theta^1}S_1^c, \tag{49c}$$

$$G_1^{P^2} = \delta\boldsymbol{A}\cdot\frac{\partial \boldsymbol{X}}{\partial \theta^2} + \partial_{\theta^2}S_1^c, \tag{49d}$$

and the second-order generating vectors related to $(\theta^1, P^1, \theta^2, P^2)$ in terms of the canonical coordinates,

$$G_2^{\theta^1} = -\left[\frac{1}{2}(G_1^{\boldsymbol{r}}\times\delta\boldsymbol{B})\cdot\frac{\partial \boldsymbol{X}}{\partial P^1} + \partial_{P^1}S_2^c\right], \tag{50a}$$

$$G_2^{\theta^2} = -\left[\frac{1}{2}(G_1^{\boldsymbol{r}}\times\delta\boldsymbol{B})\cdot\frac{\partial \boldsymbol{X}}{\partial P^2} + \partial_{P^2}S_2^c\right], \tag{50b}$$

$$G_2^{P^1} = \frac{1}{2}(G_1^{\boldsymbol{r}}\times\delta\boldsymbol{B})\cdot\frac{\partial \boldsymbol{X}}{\partial \theta^1} + \partial_{\theta^1}S_2^c, \tag{50c}$$

$$G_2^{P^2} = \frac{1}{2}(G_1^{\boldsymbol{r}}\times\delta\boldsymbol{B})\cdot\frac{\partial \boldsymbol{X}}{\partial \theta^2} + \partial_{\theta^2}S_2^c. \tag{50d}$$

Here, S_n^c is the scalar field functions in terms of the canonical coordinates. The generating vectors are determined by the GC coordinate transform $(\frac{\partial \boldsymbol{X}}{\partial P^1}, \frac{\partial \boldsymbol{X}}{\partial P^2}, \frac{\partial \boldsymbol{X}}{\partial \theta^1}, \frac{\partial \boldsymbol{X}}{\partial \theta^2})$. If the coordinate transform is known, then one can find the generating vectors.

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From Eq. (46c), the GY equations of motion related to $(\theta^1, P^1, \theta^2, P^2)$ in terms of the canonical coordinates are obtained,

$$\frac{d\bar{\theta}^1}{d\tau} = \frac{\partial(\bar{h}_0^c + \bar{h}_1^c + \bar{h}_2^c)}{\partial \bar{P}^1}, \tag{51a}$$

$$\frac{d\bar{P}^1}{d\tau} = -\frac{\partial(\bar{h}_0^c + \bar{h}_1^c + \bar{h}_2^c)}{\partial\bar{\theta}^1}, \tag{51b}$$

$$\frac{d\bar{\theta}^2}{d\tau} = \frac{\partial(\bar{h}_0^c + \bar{h}_1^c + \bar{h}_2^c)}{\partial \bar{P}^2}, \tag{51c}$$

$$\frac{d\bar{P}^2}{d\tau} = -\frac{\partial(\bar{h}_0^c + \bar{h}_1^c + \bar{h}_2^c)}{\partial\bar{\theta}^2}. \tag{51d}$$

Here, $\bar{h}_n^c(\bar{Z}^c) = [\bar{h}_n(Z(Z^c))]_{Z^c=\bar{Z}^c}$, as is indicated by Eq. (47c).

The nonlinear gyrokinetic Vlasov equation can be written in terms of the canonical coordinates as

$$\left(\partial_t + \frac{d\bar{\theta}^1}{dt}\partial_{\bar{\theta}^1} + \frac{d\bar{P}^1}{dt}\partial_{\bar{P}^1} + \frac{d\bar{\theta}^2}{dt}\partial_{\bar{\theta}^2} + \frac{d\bar{P}^2}{dt}\partial_{\bar{P}^2}\right)\bar{F}^c = 0. \tag{52}$$

Here, $\bar{F}^c(\bar{Z}^c) = [\bar{F}(Z(Z^c))]_{Z^c=\bar{Z}^c}$. θ^1 and θ^2 are two independent angle variables. For the axisymmetric tokamak, the momentum variable P_1 can be chosen as the toroidal angular momentum, a constant of the unperturbed motion. Furthermore, the momentum variable P_2 can be replaced by the energy variable U, which is also the constant of the unperturbed motion. We can use the canonical variables to compute the canonical form of the gyrocenter distribution function, $\bar{F}^c$, then transform it into the noncanonical form, which can be used to directly compute the density and current using Eq. (35).

B. The nonlinear gyrokinetic Hamilton's equations in terms of specific canonical coordinates

In this subsection, we first introduce a specific canonical variables $(\theta, P_\theta, \alpha_c, P_\alpha)$, which are defined in Refs. 11 and 32. Then, we choose the canonical variables $(\theta^1, P^1, \theta^2, P^2)$ as $(\theta, P_\theta, \alpha_c, P_\alpha)$ and develop the nonlinear canonical gyrokinetic Hamilton's equations in terms of the latter variables.

The equilibrium magnetic field in an axisymmetric tokamak can be written in terms of the magnetic coordinates (ψ, θ, ζ) as

$$\boldsymbol{B} = q(\psi)\nabla\psi \times \nabla\theta + \nabla\zeta \times \nabla\psi \tag{53a}$$

$$= g(\psi)\nabla\zeta + I(\psi)\nabla\theta + g(\psi)\delta(\psi, \theta)\nabla\psi, \tag{53b}$$

with ψ the poloidal magnetic surface. $q(\psi)$ is the safety factor. Note that the toroidal angle ζ is an ignorable coordinate. The canonical variables $(P_\alpha, \alpha_c, P_\theta, \theta)$ are expressed in terms of the noncanonical variables $(\psi, \theta, \zeta, \rho_\parallel g)$ as

$$P_\alpha = \psi - \rho_\parallel g,$$

$$P_\theta = \rho_\parallel g(q + I/g) - (\psi - \psi_0)\left(q(\psi_0) + \int_0^{\psi_0}\partial_\theta\delta d\psi\right) + \int_{\psi_0}^{\psi} q d\psi + \int_{\psi_0}^{\psi}\int_0^{\psi}\partial_\theta\delta d\psi d\psi \tag{54a}$$

$$-\rho_\parallel g\left(q - q(\psi_0) + \int_{\psi_0}^{\psi}\partial_\theta\delta d\psi\right), \tag{54b}$$

$$\alpha_c = -\zeta + q(\psi_0)\theta - \int_{\psi_0}^{\psi}\delta d\psi, \tag{54c}$$

with ψ_0 chosen as the initial value of the poloidal magnetic flux or the toroidal angular momentum. Here, $\rho_\parallel = v_\parallel/B$.

By choosing $(\theta^1, P^1, \theta^2, P^2)$ as $(\theta, P_\theta, \alpha_c, P_\alpha)$, and using Eqs. (49) and (50), we can obtain the first-order generating vectors related to $(\theta, P_\theta, \alpha_c, P_\alpha)$,

$$G_1^\theta = -\left(\delta\boldsymbol{A}\cdot\frac{\partial\boldsymbol{X}}{\partial P_\theta} + \partial_{P_\theta}S_1^c\right), \tag{55a}$$

$$G_1^{\alpha_c} = -\left(\delta\boldsymbol{A}\cdot\frac{\partial\boldsymbol{X}}{\partial P_\alpha} + \partial_{P_\alpha}S_1^c\right), \tag{55b}$$

$$G_1^{P_\theta} = \delta\boldsymbol{A}\cdot\frac{\partial\boldsymbol{X}}{\partial\theta} + \partial_\theta S_1^c, \tag{55c}$$

$$G_1^{P_\alpha} = \delta\boldsymbol{A}\cdot\frac{\partial\boldsymbol{X}}{\partial\alpha_c} + \partial_{\alpha_c}S_1^c, \tag{55d}$$

and the second-order generating vectors related to $(\theta, P_\theta, \alpha_c, P_\alpha)$,

$$G_2^\theta = -\left[\frac{1}{2}(G_1^{\boldsymbol{r}}\times\delta\boldsymbol{B})\cdot\frac{\partial\boldsymbol{X}}{\partial P_\theta} + \partial_{P_\theta}S_2^c\right], \tag{56a}$$

$$G_2^{\alpha_c} = -\left[\frac{1}{2}(G_1^{\boldsymbol{r}}\times\delta\boldsymbol{B})\cdot\frac{\partial\boldsymbol{X}}{\partial P_\alpha} + \partial_{P_\alpha}S_2^c\right], \tag{56b}$$

$$G_2^{P_\theta} = \frac{1}{2}(G_1^{\boldsymbol{r}}\times\delta\boldsymbol{B})\cdot\frac{\partial\boldsymbol{X}}{\partial\theta} + \partial_\theta S_2^c, \tag{56c}$$

$$G_2^{P_\alpha} = \frac{1}{2}(G_1^{\boldsymbol{r}}\times\delta\boldsymbol{B})\cdot\frac{\partial\boldsymbol{X}}{\partial\alpha_c} + \partial_{\alpha_c}S_2^c. \tag{56d}$$

The GC coordinate transform $(\frac{\partial\boldsymbol{X}}{\partial P^1}, \frac{\partial\boldsymbol{X}}{\partial P^2}, \frac{\partial\boldsymbol{X}}{\partial\theta^1}, \frac{\partial\boldsymbol{X}}{\partial\theta^2})$ can be found in Ref. 11. Note that Eq. (23) in Ref. 11 exactly agrees with Eq. (55).

By choosing $(\theta^1, P^1, \theta^2, P^2)$ as $(\theta, P_\theta, \alpha_c, P_\alpha)$, and using Eq. (51), the GY equations of motion related to $(\theta, P_\theta, \alpha_c, P_\alpha)$ in terms of the canonical coordinates are

$$\frac{d\bar{\theta}}{dt} = \frac{\partial\bar{h}^c}{\partial\bar{P}_\theta}, \tag{57a}$$

$$\frac{d\bar{P}_\theta}{dt} = -\frac{\partial\bar{h}^c}{\partial\bar{\theta}}, \tag{57b}$$

$$\frac{d\bar{\alpha}_c}{dt} = \frac{\partial\bar{h}^c}{\partial\bar{P}_\alpha}, \tag{57c}$$

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$$\frac{d\bar{P}_\alpha}{dt} = -\frac{\partial \bar{h}^c}{\partial \bar{\alpha}_c}, \tag{57d}$$

with

$$\bar{h}^c(\bar{Z}^c) = \bar{h}(\bar{Z}) = \bar{H}(\bar{Z}) - \bar{U} = \bar{H}_0 + \bar{H}_1 + \bar{H}_2 - \bar{U}, \tag{58a}$$

$$\bar{H}_0 = H_0, \tag{58b}$$

$$\bar{H}_1 = \langle \delta\phi \rangle - (\dot{\psi}_0 \langle \delta A_\psi \rangle + \dot{\theta}_0 \langle \delta A_\theta \rangle + \dot{\zeta}_0 \langle \delta A_\zeta \rangle + \dot{\xi}_0 \langle \delta \boldsymbol{A} \cdot \partial_\xi \boldsymbol{\rho}_0 \rangle), \tag{58c}$$

$$\bar{H}_2 = -\frac{1}{2}\langle \{ S_1, (\dot{S}_1)_0 \} \rangle + \frac{1}{2}\langle |\delta \boldsymbol{A}|^2 \rangle. \tag{58d}$$

Note that Eqs. (29c)–(29f) in Ref. 11 can be recovered from Eq. (57) in the linear approximation.

V. COMPUTATION OF THE GYROCENTER MOTION

In this section, we will give a numerical example of the code GYCAVA,[11] in which the gyrocenter equations of motion (57) are used. We will show that the second-order GY Hamiltonian (58d) is important for the trapped electron in tokamaks when the long-wavelength magnetic perturbation is large.

To model a static magnetic island, we choose the perturbation of the poloidal magnetic flux as

$$\psi_1 = -\delta A_\zeta = \psi_{1(2,1)}(\psi)\cos(2\theta - \zeta), \tag{59a}$$

$$\psi_{1(2,1)}(\psi) = \epsilon_\psi(\psi - \psi_{axis})(\psi_b - \psi). \tag{59b}$$

Here, ϵ_ψ is a small constant. $(\psi_{axis} = 0, \psi_b = 0.098Wb)$ are the values of the equilibrium poloidal magnetic flux at the magnetic axis and the boundary of the tokamak, respectively. The parameters of the perturbation are chosen as $\psi_{1(2,1)}(\psi_s)/\psi_s = 10^{-3}$, with ψ_s the value of the equilibrium poloidal magnetic flux on the rational surface where $q = 2$. With the perturbation, the ratio of the width of the magnetic island to the minor radius is about 6%. The scale length of the imposed perturbation of the poloidal magnetic flux has the same order as the minor radius of the tokamak, that is, $k_\perp \rho_0 \ll 1$ with $k_\perp$ the wave number in the perpendicular direction. Thus, the finite-Larmor-radius effect is not important in this numerical example. Furthermore, the ratio of the first term of the right-hand side of Eq. (58d) to the second term has the order of $O(k_\perp^2 \rho_0^2)$. Thus, the first term can be neglected in contrast to the second term in this numerical example.

The Poincare section plots for the trapped electron orbits without and with the second-order GY Hamiltonian (58d) are shown in Figs. 1(a) and 1(b). The banana width of the trapped electron with $\bar{H}_2$ is small, while the one without $\bar{H}_2$ is large.

The deviations of Hamiltonian and the longitudinal invariant are shown in Fig. 2 for the trapped electrons, which are defined as

$$\Delta E/E_0 \equiv \frac{\bar{H}(\bar{Z}) - [\bar{H}(\bar{Z})]_{t=0}}{[\bar{H}(\bar{Z})]_{t=0}}, \tag{60a}$$

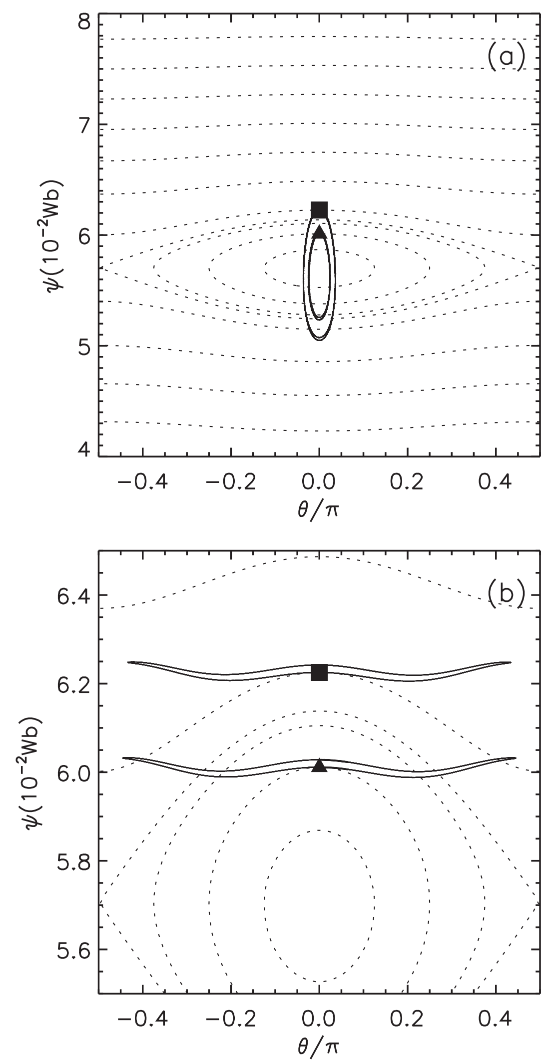

FIG. 1. Poincare section plots of the magnetic flux surfaces (dotted line) and the trapped electron orbits (dotted symbol) with the initial energy $E_0 = 1\,$keV and the pitch $v_\parallel/v = 0.4$. Two different launch points are labeled by the square and the triangle symbols, respectively. (a) and (b) are for the orbits without and with $\bar{H}_2$, respectively.

$$\Delta \mathcal{J}/\mathcal{J}_0 \equiv \frac{\mathcal{J} - \mathcal{J}_0}{\mathcal{J}_0}. \tag{60b}$$

Here, $\bar{H}(\bar{Z})$ and $\mathcal{J}$ are the gyrocenter Hamiltonian function evaluated at the gyrocenter coordinates and the longitudinal invariant, respectively; $[\bar{H}(\bar{Z})]_{t=0}$ and $\mathcal{J}_0$ are the initial values of $\bar{H}(\bar{Z})$ and $\mathcal{J}$, respectively. The longitudinal invariant is defined as $\mathcal{J} = \oint P_\theta d\theta$.[11,32]

The initial error of the energy defined as $(\bar{H}(\bar{Z})_{t=0} - E_0)/E_0$ is -0.07 for the orbit without $\bar{H}_2$ and 3×10^{-4} for the orbit with $\bar{H}_2$. It indicates that $\bar{H}_2(\bar{Z})_{t=0}/E_0 = 0.07$.

Note that the particle energy conserves for a static magnetic island. It is seen from Fig. 2 that the deviations of Hamiltonian have the order of 10^{-3} for the orbit without $\bar{H}_2$ and the order of 10^{-4} for the orbit with $\bar{H}_2$, which numerically demonstrate the conservation of the particle energy.

From Fig. 2, it is seen that the deviations of the longitudinal invariant have the order of 10^{-1} for the orbit without $\bar{H}_2$ and the order of 10^{-3} for the orbit with $\bar{H}_2$. It indicates that $\mathcal{J}$ is still a good invariant with $\bar{H}_2$, while $\mathcal{J}$ is not a

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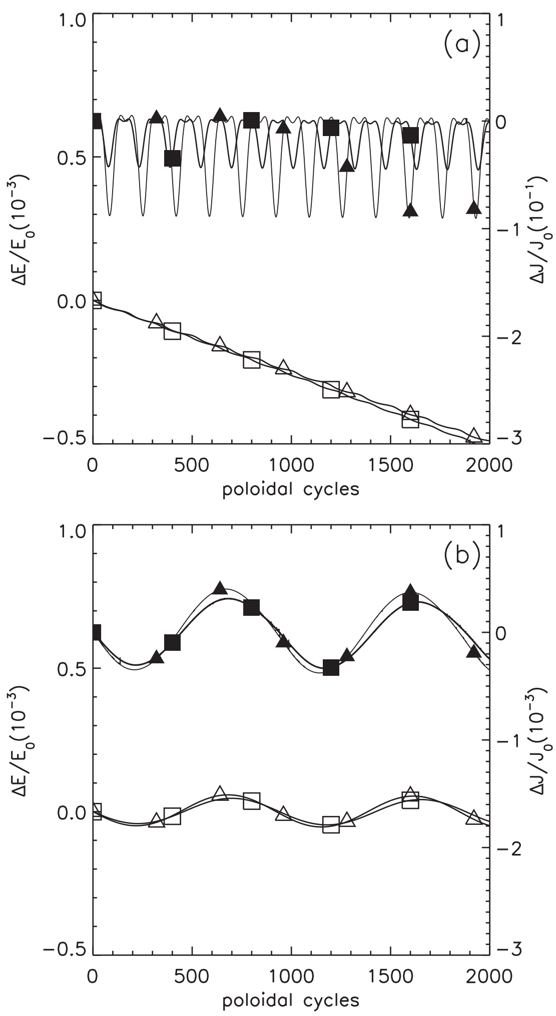

FIG. 2. The deviations of Hamiltonian $\Delta E/E_0$ (labeled by "Δ" and "□") and the longitudinal invariant $\Delta\mathcal{J}/\mathcal{J}_0$ (labeled by "▲" and "■") for trapped electrons shown in Fig. 1. (a) and (b) are for the orbits without and with $\bar{H}_2$, respectively.

good invariant without $\bar{H}_2$. The underlying physics of the longitudinal invariant $\mathcal{J}$ for the trapped particles with a magnetic perturbation was discussed in Ref. 11.

According to the above discussion, we know that the trapped electron orbits without $\bar{H}_2$ are unphysical due to the large numerical error produced by neglecting $\bar{H}_2$. Therefore, the second-order GY Hamiltonian is important for the GY motion of the trapped electron in tokamaks with a large magnetic perturbation.

VI. SUMMARY

The nonlinear gyrokinetic Vlasov equation in terms of the canonical coordinates has been derived using the property of the coordinate transform, instead of carrying out the Lie-transform perturbation calculation. In the linear approximation, the results exactly recover the previous linear canonical gyrokinetic equations[11] derived by the Lie-transform perturbation method.

The computation of the GY motion with a large magnetic perturbation has been presented and discussed. The numerical results indicate that the second-order GY Hamiltonian (58d) is important for the GY motion of the trapped electron in tokamaks with a large magnetic perturbation.

For a new set of GC coordinates and its corresponding GY coordinates (canonical or noncanonical), we do not need to derive the gyrokinetic theory by carrying out the Lie-transform perturbation calculation again. If we know the gyrokinetic equations based on one set of GC coordinates, then we can directly obtain the gyrokinetic equations based on another set of GC coordinates using the property of the coordinate transform.

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[1]T. M. Antonsen, Jr. and B. Lane, Phys. Fluids **23**, 1205 (1980).
[2]P. J. Catto, W. M. Tang, and D. E. Baldwin, Plasma Phys. **23**, 639 (1981).
[3]E. A. Frieman and L. Chen, Phys. Fluids **25**, 502 (1982).
[4]D. H. E. Dubin, J. A. Krommes, C. Oberman, and W. W. Lee, Phys. Fluids **26**, 3524 (1983).
[5]T. S. Hahm, W. W. Lee, and A. J. Brizard, Phys. Fluids **31**, 1940 (1988).
[6]T. S. Hahm, Phys. Fluids **31**, 2670 (1988).
[7]A. J. Brizard, J. Plasma Phys. **41**, 541 (1989).
[8]A. J. Brizard and T. S. Hahm, Rev. Mod. Phys. **79**, 421 (2007).
[9]S. Wang, Phys. Rev. E **64**, 056404 (2001).
[10]L. Qi and S. Wang, Phys. Plasmas **16**, 062504 (2009).
[11]Y. Xu, X. Xiao, and S. Wang, Phys. Plasmas **18**, 042505 (2011).
[12]S. Wang, Phys. Plasmas **19**, 062504 (2012).
[13]W. W. Lee, Phys. Fluids **26**, 556 (1983).
[14]S. E. Parker and W. W. Lee, Phys. Fluids B **5**, 77 (1993).
[15]M. Kotschenreuther, G. Rewoldt, and W. M. Tang, Comput. Phys. Commun. **88**, 128 (1995).
[16]Z. Lin, T. S. Hahm, W. W. Lee, W. M. Tang, and R. B. White, Science **281**, 1835 (1998).
[17]F. Jenko, W. Dorland, M. Kotschenreuther, and B. N. Rogers, Phys. Plasmas **7**, 1904 (2000).
[18]J. Candy and R. E. Waltz, J. Comput. Phys. **186**, 545 (2003).
[19]X. Q. Xu, Z. Xiong, M. R. Dorr, J. A. Hittinger, K. Bodi, J. Candy, B. I. Cohen, R. H. Cohen, P. Colella, G. D. Kerbel, S. Krasheninnikov, W. M. Nevins, H. Qin, T. D. Rognlien, P. B. Snyder, and M. V. Umansky, Nucl. Fusion **47**, 809 (2007).
[20]X. Garbet, Y. Idomura, L. Villard, and T. H. Watanabe, Nucl. Fusion **50**, 043002 (2010).
[21]R. G. Littlejohn, J. Math. Phys. **20**, 2445 (1979).
[22]J. R. Cary, Phys. Rep. **79**, 129 (1981).
[23]R. G. Littlejohn, J. Math. Phys. **23**, 742 (1982).
[24]J. R. Cary and R. G. Littlejohn, Ann. Phys. (N.Y.) **151**, 1 (1983).
[25]R. G. Littlejohn, Phys. Fluids **24**, 1730 (1981).
[26]R. G. Littlejohn, J. Plasma Phys. **29**, 111 (1983).
[27]J. R. Cary and A. J. Brizard, Rev. Mod. Phys. **81**, 693 (2009).
[28]X. Xiao and S. Wang, Phys. Plasmas **15**, 122511 (2008).
[29]Y. Idomura, S. Tokuda, and Y. Kishimoto, Nucl. Fusion **43**, 234 (2003).
[30]P. Angelino, A. Bottino, R. Hatzky, S. Jolliet, O. Sauter, T. M. Tran, and L. Villard, Phys. Plasmas **13**, 052304 (2006).
[31]H. Goldstein, C. Poole, and J. Safko, *Classical Mechanics* (Addison-Wesley, 2002), Chap. 9, p. 392.
[32]S. Wang, Phys. Plasmas **13**, 52506 (2006).

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