Nonlinear gyrokinetic theory based on a new method and computation of the guiding-center orbit in tokamaks

Yingfeng Xu, Zongliang Dai, and Shaojie Wang

Citation: Physics of Plasmas (1994-present) **21**, 042505 (2014); doi: 10.1063/1.4871726
View online: http://dx.doi.org/10.1063/1.4871726
View Table of Contents: http://scitation.aip.org/content/aip/journal/pop/21/4?ver=pdfcov
Published by the AIP Publishing

PHYSICS OF PLASMAS **21**, 042505 (2014)

Nonlinear gyrokinetic theory based on a new method and computation of the guiding-center orbit in tokamaks

Yingfeng Xu,[a)] Zongliang Dai, and Shaojie Wang
Department of Modern Physics, University of Science and Technology of China, Hefei 230026, China

(Received 24 January 2014; accepted 28 March 2014; published online 18 April 2014)

The nonlinear gyrokinetic theory in the tokamak configuration based on the two-step transform is developed; in the first step, we transform the magnetic potential perturbation to the Hamiltonian part, and in the second step, we transform away the gyroangle-dependent part of the perturbed Hamiltonian. Then the I-transform method is used to decoupled the perturbation part of the motion from the unperturbed motion. The application of the I-transform method to the computation of the guiding-center orbit and the guiding-center distribution function in tokamaks is presented. It is demonstrated that the I-transform method of the orbit computation which involves integrating only along the unperturbed orbit agrees with the conventional method which integrates along the full orbit. A numerical code based on the I-transform method is developed and two numerical examples are given to verify the new method. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4871726]

I. INTRODUCTION

The turbulence transport is a very important topic in the tokamak confinement research. The micro-instabilities related to the turbulence are often investigated by using the gyrokinetic theory[1–8] and simulation.[9–16] In the modern gyrokinetic theory,[4–8,17–23] the Lie-transform perturbation method[24–27] is used to decouple the gyrocenter motion from the gyromotion. Then the kinetic equation can be reduced to the gyrokinetic equation, which is much easier to investigate the evolution of the tokamak plasmas on the timescale longer than the gyro-period.

The Lie-transform method is a very effective and general method to treat the perturbation problem. Several different transform procedures have been used in nonlinear gyrokinetic theories.[5,7,8] In the gyrokinetic theory developed by Brizard and Hahm,[7,8] one-step transform procedure from the guiding-center coordinates to the gyrocenter coordinates is used to derive the gyrokinetic equations. The gyrocenter transform of the procedure includes the $\delta \boldsymbol{A}$ term and the S_n term. Here, $\delta \boldsymbol{A}$ is the perturbation part of the magnetic potential. S_n is the nth-order gauge function, used to remove the gyroangle-dependent part the Lagrangian and thus to keep the gyrocenter magnetic moment conservative. The gyrokinetic equations developed by Hahm *et al*.[5] have been widely used in the electromagnetic gyrokinetic simulation.[16] His Lie-transform perturbation method is based on a two-step transform procedure. The first step is to move the $\delta A_{\parallel}$ term into the Hamiltonian part by a simple transform, which makes all the perturbations appear in the Hamiltonian part and keep the transformed Poisson bracket formally same as the unperturbed one. The second step is the conventional Lie-transform determined by the perturbations only through the gauge function S_n. However, the perpendicular component of the magnetic potential perturbation is not included and the slab geometry is used in his work. This two-step transform procedure also has been used to derive the kinetic theory of turbulence by Wang,[22] and the full electromagnetic potential perturbations are included. In this kinetic theory, the first-step transform procedure, which is related to the magnetic potential perturbation $\delta \boldsymbol{A}$, is clear and easy to understand and may be useful in the gyrokinetic simulation. The first-step transform procedure makes the transformed Poisson bracket independent of perturbations.

Recently, the I-transform method is developed, which can be used to decoupled the perturbation part of the motion from the unperturbed motion, is a useful method in the kinetic and gyrokinetic theory.[21–23] The method has been developed to formulate the nonlinear gyrokinetic equation in the Fokker-Planck form and to find the nonlinear scattering term from the nonlinear gyrokinetic equation. Note that the first-order I-transform was introduced by Cary and Kaufman[28] in discussing the ponderomotive effects on the dynamics of the oscillation center.

In this paper, first we derive the nonlinear gyrokinetic equations in the tokamak configuration including the full electromagnetic potential perturbations by the two-step transform procedure from the guiding-center coordinates to the gyrocenter coordinates. Then the I-transform method is used to make the transformed gyrokinetic equations same as the unperturbed ones. Finally, a numerical orbit code based on the short-time I-transform method is developed to compute the guiding-center orbit, and two numerical examples are given to validate the new code.

The remaining part of this paper is organized as follows. In Sec. II, the guiding-center theory is reviewed. In Sec. III, the nonlinear gyrokinetic theory based on the two-step transform procedure is developed. In Sec. IV, the gyrokinetic equation based on the I-transform method is derived and its application to the computation of the guiding-center orbit and the guiding-center distribution function is presented. In Sec. V, we develop a numerical orbit code based on the short-time I-transform method and give two numerical examples to validate the new method. In Sec. VI, the main results are summarized.

[a)]Electronic mail: yfengxu@mail.ustc.edu.cn

II. REVIEW OF THE UNPERTURBED GUIDING-CENTER THEORY

In the guiding-center theory, the guiding-center transform, which can be solved by the Lie-transform perturbation method, is used to remove the fast gyromotion.[29] By using the Lie-transform perturbation method, the guiding-center transformation from the particle phase-space coordinates $(\boldsymbol{r}, v_{\|p}, \mu_p, \xi_p)$ to the guiding-center phase-space coordinates $(\boldsymbol{X}, v_{\|}, \mu, \xi)$ is expanded in powers of a small parameter ϵ_B. Here, $\epsilon_B = \rho_0/L$, where ρ_0 is the Larmor radius, and L is the characteristic length of the equilibrium magnetic field B_0. $\boldsymbol{X}$ is the guiding-center coordinates, $v_{\|}$ is the velocity component parallel to the unperturbed magnetic field $\boldsymbol{B}_0$, μ is the magnetic moment, and ξ is the gyroangle. The subscript p denotes the phase-space coordinates of particle. In general guiding-center theory, the guiding-center transform $\boldsymbol{X} = \boldsymbol{r} - \boldsymbol{\rho}_0$, $v_{\|} = v_{\|p}$, $\mu = \mu_p$, and $\xi = \xi_p$ are often used. The unperturbed fundamental one-form (the guiding-center Lagrangian) for a charged particle,

$$\hat{\Gamma}_0 \equiv \Gamma_0 - H_0 dt = \Gamma_{0i} dZ^i - H_0 dt, \tag{1}$$

can be written in terms of the noncanonical guiding-center coordinates.[30–32] The unperturbed symplectic part and the unperturbed guiding-center Hamiltonian are

$$\Gamma_0 = (m v_{\|} \boldsymbol{b}_0 + m \boldsymbol{u}_E + e \boldsymbol{A}_0) \cdot d\boldsymbol{X} + \frac{m}{e} \mu d\xi, \tag{2a}$$

$$H_0 = \frac{1}{2} m (v_{\|}^2 + u_E^2) + \mu B_0 + \frac{m\mu}{e} \boldsymbol{b}_0 \cdot \nabla \times \boldsymbol{u}_E + e\phi_0. \tag{2b}$$

Here, m and e are the mass and the charge of the particle, respectively. The equilibrium $E \times B$ velocity $\boldsymbol{u}_E \equiv \boldsymbol{E}_0 \times \boldsymbol{b}_0 / B_0$ associated with the equilibrium ϕ_0 can be chosen of the order of the thermal velocity, that is, $u_E/v_{th} \sim 1$.[31,32]

The Lagrange two-form is defined as $\omega \equiv d\Gamma$.[8,33] The unperturbed Lagrange two-form can be written as $\hat{\omega}_0 \equiv \omega_0 - dH_0 \wedge dt$. The two-form ω_0 is

$$\omega_0 = \frac{1}{2} \epsilon_{ijk} e B_0^{*k} dX^i \wedge dX^j + m b_{0j} dv_{\|} \wedge dX^j + (m/e) d\mu \wedge d\xi - dH_0 \wedge dt, \tag{3}$$

where

$$\boldsymbol{B}_0^* = \boldsymbol{B}_0 + (m/e) v_{\|} \nabla \times \boldsymbol{b}_0 + (m/e) \nabla \times \boldsymbol{u}_E, \tag{4}$$

d is the exterior differential, $\wedge$ is the exterior product, ϵ_{ijk} is the permutation tensor, and b_{0j} is the component of the unit vector $\boldsymbol{b}_0$ along the unperturbed magnetic field.

The unperturbed guiding-center equations of motion are written as

$$\dot{Z}_0^i \equiv \frac{d_0 Z^i}{dt} = \{Z^i, H_0\} = J_0^{ij} \partial_j H_0. \tag{5}$$

Here, $\boldsymbol{J}_0$ is the unperturbed Poisson matrix, which is the inverse matrix of the unperturbed Lagrange matrix $\boldsymbol{\omega}_0$.[8,33] The non-zero components of the unperturbed Poisson matrix $\boldsymbol{J}_0$ are

$$J_0^{X^i X^j} = -\frac{\epsilon^{ijk} b_{0k}}{e B_{\|0}^*}, \tag{6a}$$

$$J_0^{X^i v_{\|}} = -J_0^{v_{\|} X^i} = \frac{B_0^{*i}}{m B_{\|0}^*}, \tag{6b}$$

$$J_0^{\xi\mu} = -J_0^{\mu\xi} = e/m. \tag{6c}$$

Here, $B_{\|0}^* = \boldsymbol{B}_0^* \cdot \boldsymbol{b}_0$.

III. NONLINEAR GYROCENTER EQUATIONS OF MOTION AND GYROKINETIC VLASOV-MAXWELL EQUATIONS

In the standard modern nonlinear gyrokinetic theory,[8] the amplitude of the perturbations is much smaller than the corresponding equilibrium quantities, that is, $\frac{|\delta \boldsymbol{E}|}{v_{th} B_0} \sim \frac{|\delta \boldsymbol{B}|}{B_0} \sim \frac{\delta f}{f_0} \sim \epsilon_\delta \ll 1$. Here, $(\delta\boldsymbol{E}, \delta\boldsymbol{B})$ are the perturbation parts of the electromagnetic fields defined as $\delta\boldsymbol{B} = \nabla \times \delta\boldsymbol{A}$, $\delta\boldsymbol{E} = -\nabla\delta\phi - \partial_t \delta\boldsymbol{A}$, and v_{th} is the thermal velocity of the particle. δf and f_0 are the perturbation part and the unperturbed part of the distribution function, respectively. The frequency ω and perpendicular wave vector $\boldsymbol{k}_\perp$ of the perturbations satisfy the conditions, $\omega/\Omega \ll 1, k_\perp \rho_0 \sim 1$. Here, Ω is the gyro-frequency of a charged particle.

With the electromagnetic perturbation potentials $(\delta\phi, \delta\boldsymbol{A})$ introduced, the conservation of the guiding-center magnetic moment μ is broken. For removing the fast gyromotion, the gyrocenter conservative magnetic moment $\bar{\mu}$ is found by the gyrocenter transformation from the guiding-center coordinates to the gyrocenter coordinates.

The fundamental one-form can be separated into the unperturbed part and the perturbation part

$$\hat{\Gamma} \equiv \hat{\Gamma}_0 + \hat{\Gamma}_1, \tag{7a}$$

$$\hat{\Gamma}_1 = \Gamma_{1i} dZ^i - H_1 dt = e\delta\boldsymbol{A}(\boldsymbol{X} + \boldsymbol{\rho}_0, t) \cdot d(\boldsymbol{X} + \boldsymbol{\rho}_0) - e\delta\phi(\boldsymbol{X} + \boldsymbol{\rho}_0, t) dt. \tag{7b}$$

The first-order Lagrange two-form $\hat{\omega}_1$ can be rewritten as

$$\begin{aligned}\hat{\omega}_1 &= d\hat{\Gamma}_1 = \omega_1 - dH_1 \wedge dt \\ &= \frac{1}{2} \epsilon_{ijk} e \delta B^k dX^i \wedge dX^j + e\delta E_j dX^j \wedge dt \\ &\quad + \epsilon_{ijk} e \delta B^k \partial_\mu \rho_0^i d\mu \wedge dX^j \\ &\quad + \epsilon_{ijk} e \delta B^k \partial_\xi \rho_0^i d\xi \wedge dX^j \\ &\quad + \epsilon_{ijk} e \partial_\mu \rho_0^i \partial_\xi \rho_0^j \delta B^k d\mu \wedge d\xi \\ &\quad + e\delta\boldsymbol{E} \cdot \partial_\mu \boldsymbol{\rho}_0 d\mu \wedge dt + e\delta\boldsymbol{E} \cdot \partial_\xi \boldsymbol{\rho}_0 d\xi \wedge dt,\end{aligned} \tag{8}$$

with the first-order Lagrange two-forms ω_1 defined as $\omega_1 \equiv d\Gamma_1$, that is, $\omega_{1ji} = \partial_j \Gamma_{1i} - \partial_i \Gamma_{1j}$. Hereafter, the spatial dependence of the gyroradius vector $\boldsymbol{\rho}_0$ is ignored in the following, that is, $\boldsymbol{\rho}_0 = \boldsymbol{\rho}_0(\mu, \xi)$.[8]

In the Subsections III A–III C, first we move the perturbation part of magnetic potential $\delta\boldsymbol{A}$ into the Hamiltonian part by the Lie-transform perturbation method. This transform makes the symplectic part of the transformed fundamental one-form same as the unperturbed one. We call it $\delta\boldsymbol{A}$

transform. Then the conventional Lie-transform method is used to remove the gyroangle-dependent part of the Hamiltonian by the gauge function S_n. Finally, the gyrokinetic equations are presented.

A. The δA transform

The $\delta\boldsymbol{A}$ transform $\mathcal{T}_A$ from the guiding-center coordinates Z to the transformed coordinates Z^* is expressed in terms of $\delta\boldsymbol{g}$, that is $Z^{*i} = \mathcal{T}_A Z^i$. According to the conventional Lie-transform perturbation method, $\delta\boldsymbol{A}$ transform can be expanded in powers of the amplitude ordering parameter ϵ_δ, written as

$$Z^{*i} = Z^i + \delta g_1^i + \delta g_2^i + \frac{1}{2}\delta g_1^j \partial_j \delta g_1^i + \ldots, \tag{9}$$

where $\delta\boldsymbol{g}_1$ and $\delta\boldsymbol{g}_2$ are the first-order and second-order generating vector fields, respectively.

The transformed fundamental one-form is defined as $\hat{\Gamma}^* = \Gamma^* - H^* dt = \Gamma_i^* d\bar{Z}^{*i} - H^* dt$. To make the symplectic part of the transformed fundamental one-form same as the unperturbed one, we set $\Gamma_i^* = \Gamma_{0i}$. Then, we have $\hat{\Gamma}^* = \Gamma_0 - H^* dt$. From the symplectic part of $\hat{\Gamma}^* = \mathbb{T}_A^{-1}\hat{\Gamma}$, we obtain

$$0 = \Gamma_{1i} - \delta g_1^j \omega_{0ji}, \tag{10a}$$

$$0 = -\delta g_n^j \omega_{0ji} - \delta g_{n-1}^j \omega_{1ji}, \quad for\, n \geq 2. \tag{10b}$$

Note that

$$\{\boldsymbol{X} + \boldsymbol{\rho}_0, \boldsymbol{X} + \boldsymbol{\rho}_0\} = 0. \tag{11}$$

Using Eq. (11), we can obtain the nth-order generating vectors

$$\delta g_1^i = e\delta\boldsymbol{A} \cdot \partial_j(\boldsymbol{X} + \boldsymbol{\rho}_0) J_0^{ji}, \tag{12a}$$

$$\delta g_n^i = 0, \quad for\, n \geq 2. \tag{12b}$$

Then, the $\delta\boldsymbol{A}$ transform can be simplified as

$$Z^{*i} = Z^i + \delta g_1^i + \frac{1}{2}\delta g_1^j \partial_j \delta g_1^i. \tag{13}$$

It is not hard to show that $\delta\boldsymbol{g}_1$ is an incompressible flow in the phase space

$$\frac{1}{\mathcal{J}}\partial_i(\mathcal{J}\delta g_1^i) = 0, \tag{14}$$

with $\mathcal{J}$ being the Jacobian of the phase space. The components of $\delta\boldsymbol{g}$ can be explicitly written as

$$\delta g_1^{\boldsymbol{X}} = -\frac{\boldsymbol{b}_0}{B_{\|0}^*} \times \delta\boldsymbol{A}, \tag{15a}$$

$$\delta g_1^{v_\|} = \frac{e\boldsymbol{B}_0^*}{mB_{\|0}^*} \cdot \delta\boldsymbol{A}, \tag{15b}$$

$$\delta g_1^{\xi} = -\frac{e^2}{m}\delta\boldsymbol{A} \cdot \partial_\mu \boldsymbol{\rho}_0, \tag{15c}$$

$$\delta g_1^{\mu} = \frac{e^2}{m}\delta\boldsymbol{A} \cdot \partial_\xi \boldsymbol{\rho}_0. \tag{15d}$$

The transformed Hamiltonian can be obtained by the pull-back transform

$$H^* = \mathbb{T}_A^{-1} H = H - \delta g_1^i \partial_i H + \frac{1}{2}\delta g_1^j \partial_j(\delta g_1^i \partial_i H), \tag{16}$$

with $H = H_0 + H_1$.

After the $\delta\boldsymbol{A}$ transform, we have the transformed fundamental one-form

$$\hat{\Gamma}^* = \Gamma_0 - (H_0 + \delta H^*)dt, \tag{17a}$$

$$\delta H^* = e\delta\phi - e(\dot{\boldsymbol{X}}_0 + \dot{\xi}_0 \partial_\xi \boldsymbol{\rho}_0) \cdot \delta\boldsymbol{A} + \frac{e^2}{2m}|\delta\boldsymbol{A}|^2. \tag{17b}$$

Note that all the perturbations appear in the Hamiltonian part.

B. The gyrokinetic Lie transform

Note that the perturbed Hamiltonian in Eq. (17b) is gyroangle-dependent. To decouple the gyromotion from the gyrocenter motion, we need to make the gyrocenter Lie-transform.

The gyrocenter phase-space transformation $\mathcal{T}_S$ from the variables Z^* to the gyrocenter variables $\bar{Z}$ defined as $\bar{Z}^i = \mathcal{T}_S Z^{*i}$ can be expanded in powers of the amplitude ordering parameter ϵ_δ up to $O(\epsilon_\delta^2)$ written as

$$\bar{Z}^i = Z^{*i} + G_1^i + G_2^i + \frac{1}{2}G_1^j \partial_j G_1^i, \tag{18}$$

where $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ are the first-order and second-order generating vector fields, respectively.

The transformed fundamental one-form is defined as

$$\hat{\bar{\Gamma}} \equiv \bar{\Gamma} - \bar{H}dt = \bar{\Gamma}_i d\bar{Z}^i - \bar{H}dt. \tag{19}$$

We choose $\bar{\Gamma}_i = \Gamma_{i0}$, that is, the transformed Lagrangian two-form is formally same as the unperturbed one, then $\bar{\Gamma}_{ni} = 0$, $\bar{\omega}_1 = 0$, $\bar{\omega} = \omega_0$. By using this choice, the transformation of the fundamental one-form, $\hat{\bar{\Gamma}} = \mathbb{T}_S^{-1}\hat{\Gamma}^* + dS$, can be expanded up to $O(\epsilon_\delta^2)$,

$$\bar{\Gamma}_{0i} = \Gamma_{0i}, \tag{20a}$$

$$0 = -G_n^j \omega_{0ji} + \partial_i S_n, \quad (n = 1, 2), \tag{20b}$$

$$\bar{H}_0 = H_0, \tag{20c}$$

$$\bar{H}_1 = \delta H^* - G_1^i \partial_i H_0, \tag{20d}$$

$$\bar{H}_2 = -G_1^i \partial_i \delta H^* + \frac{1}{2}G_1^i \partial_i(G_1^j \partial_j H_0) - G_2^i \partial_i H_0. \tag{20e}$$

Here, $\mathbb{T}_S^{-1}$ is the push-forward transformation. S_1 and S_2 are the first-order and second-order scalar field function used for removing the gyroangle dependence of the Hamiltonian.

From Eq. (20b), we can find the generating vector fields,

$$G_n^i = \partial_j S_n J_0^{ji}, \quad (n = 1, 2). \tag{21}$$

The components of the nth-order generating vector field can be explicitly written in terms of the noncanonical variables in the generic form[21]

$$G_n^{\boldsymbol{X}} = -\frac{\boldsymbol{b}_0}{eB^*_{\parallel 0}} \times \nabla S_n - \partial_{v_\parallel} S_n \frac{\boldsymbol{B}_0^*}{mB^*_{\parallel 0}}, \tag{22a}$$

$$G_n^{v_\parallel} = \frac{\boldsymbol{B}_0^*}{mB^*_{\parallel 0}} \cdot \nabla S_n, \tag{22b}$$

$$G_n^{\xi} = -\frac{e}{m}\partial_\mu S_n, \tag{22c}$$

$$G_n^{\mu} = \frac{e}{m}\partial_\xi S_n. \tag{22d}$$

From Eq. (21), we can see that $\boldsymbol{G}_n$'s are the Hamiltonian flows. Therefore, $\boldsymbol{G}_n$'s are incompressible flows in the phase space, that is,

$$\frac{1}{\mathcal{J}}\partial_i(\mathcal{J}G_n^i) = 0. \tag{23}$$

Here, $\mathcal{J}$ is the Jacobian of the phase space.

To decouple the gyrocenter motion from the gyromotion, the gyrocenter Hamiltonians $\bar{H}_n$ are chosen to satisfy the condition $\bar{H}_n = \langle \bar{H}_n \rangle$. Here, $\langle \cdots \rangle$ denotes the gyroaverage. Thus, the first-order and second-order scalar field functions can be chosen as

$$\frac{d_0 S_1}{dt} = \widetilde{\delta H^*}, \tag{24a}$$

$$\frac{d_0 S_2}{dt} = -\frac{1}{2}\widetilde{\{S_1, (\dot{S}_1)_0\}} - \{S_1, \langle \delta H^* \rangle\}, \tag{24b}$$

with $\frac{d_0}{dt}$ defined as $\frac{d_0}{dt} = \partial_t + \dot{\boldsymbol{X}}_0 \cdot \nabla + \dot{v}_{\parallel 0}\partial_{v_\parallel} + \dot{\xi}_0 \partial_\xi$. For any function A, $\tilde{A}$ denotes the gyroangle dependence part, defined as $\tilde{A} \equiv A - \langle A \rangle$. By substituting Eq. (24) into Eqs. (20d) and (20e), the first-order and the second-order gyrocenter Hamiltonians are written as

$$\begin{aligned} \bar{H}_1 = \langle \delta H^* \rangle &= e\langle \delta\phi \rangle - e\dot{\boldsymbol{X}}_0 \cdot \langle \delta \boldsymbol{A} \rangle \\ &- e\dot{\xi}_0 \langle \delta\boldsymbol{A} \cdot \partial_\xi \boldsymbol{\rho}_0 \rangle + \frac{e^2}{2m}\langle |\delta \boldsymbol{A}|^2 \rangle, \end{aligned} \tag{25a}$$

$$\bar{H}_2 = -\frac{1}{2}\langle \{S_1, (\dot{S}_1)_0\} \rangle. \tag{25b}$$

C. Equations of the gyrocenter motion and Maxwell-Vlasov equations

The gyrocenter equations of motion are written in terms of the Poisson matrix as

$$\frac{d\bar{Z}^i}{dt} = J_0^{ij}\partial_j \bar{H} = J_0^{ij}\partial_j(\bar{H}_0 + \bar{H}_1 + \bar{H}_2). \tag{26}$$

Note that the gyrocenter equations of motion satisfy the Liouville's theorem.[8] The nonlinear gyrokinetic Vlasov equation $\partial_t \bar{F} + \{\bar{F}, \bar{H}\} = 0$ can be written in terms of the noncanonical coordinates as

$$\partial_t \bar{F} + \frac{d\bar{\boldsymbol{X}}}{dt} \cdot \bar{\nabla}\bar{F} + \frac{d\bar{v}_\parallel}{dt}\partial_{\bar{v}_\parallel}\bar{F} = 0. \tag{27}$$

The guiding-center distribution function can be obtained by the two-step pull-back transform of the gyrocenter distribution function, that is, $F = \mathbb{T}_A F^* = \mathbb{T}_A \mathbb{T}_S \bar{F}$

$$F^* = \mathbb{T}_S \bar{F} = \bar{F} + G_1^i \partial_i \bar{F} + G_2^i \partial_i \bar{F} + \frac{1}{2}G_1^i \partial_i (G_1^j \partial_j \bar{F}), \tag{28a}$$

$$F = \mathbb{T}_A F^* = F^* + \delta g_1^i \partial_i F^* + \frac{1}{2}\delta g_1^j \partial_j (\delta g_1^i \partial_i F^*). \tag{28b}$$

The Poisson equation and Ampere law can be expressed as

$$\nabla^2(\phi_0 + \delta\phi) = -\frac{1}{\epsilon_0}\sum \rho, \tag{29a}$$

$$\nabla^2(\boldsymbol{A}_0 + \delta\boldsymbol{A}) = -\mu_0 \sum \boldsymbol{J}. \tag{29b}$$

Here, the displacement current has been dropped for the low-frequency perturbations, and the Coulomb gauge $\nabla \cdot (\boldsymbol{A}_0 + \delta \boldsymbol{A}) = 0$ has been used. The charge density and the current are expressed in terms of the coordinates Z^*, written as

$$\rho = e\int F^* \delta(\boldsymbol{X}^* + \boldsymbol{\rho}_0 - \boldsymbol{r}) d^6\boldsymbol{Z}^*, \tag{30a}$$

$$\boldsymbol{J} = e\int \mathbb{T}_A^{-1}(\dot{\boldsymbol{X}}_0 + \dot{\xi}_0 \partial_\xi \boldsymbol{\rho}_0) F^* \delta(\boldsymbol{X}^* + \boldsymbol{\rho}_0 - \boldsymbol{r}) d^6\boldsymbol{Z}^*, \tag{30b}$$

with $d^6\boldsymbol{Z}^* = \frac{B^*_{\parallel 0}}{m} d^3\boldsymbol{X}^* dv_\parallel^* d\mu^* d\xi^*$. Here, $\boldsymbol{X} + \boldsymbol{\rho}_0(Z) = \boldsymbol{X}^* + \boldsymbol{\rho}_0(Z^*)$ has been used.

Using Eq. (15), we can obtain $\mathbb{T}_A^{-1}(\dot{\boldsymbol{X}}_0 + \dot{\xi}_0 \partial_\xi \boldsymbol{\rho}_0) = \dot{\boldsymbol{X}}_0 + \dot{\xi}_0 \partial_\xi \boldsymbol{\rho}_0 - \frac{e}{m}\delta\boldsymbol{A}$. Then the expressions of the density and the current can be rewritten as

$$\rho = e\int \mathbb{T}_S \bar{F} \delta(\boldsymbol{X}^* + \boldsymbol{\rho}_0 - \boldsymbol{r}) d^6\boldsymbol{Z}^*, \tag{31a}$$

$$\boldsymbol{J} = e\int (\dot{\boldsymbol{X}}_0 + \dot{\xi}_0 \partial_\xi \boldsymbol{\rho}_0) \mathbb{T}_S \bar{F} \delta(\boldsymbol{X}^* + \boldsymbol{\rho}_0 - \boldsymbol{r}) d^6\boldsymbol{Z}^* - \frac{\omega_p^2}{\mu_0 c^2}\delta\boldsymbol{A}. \tag{31b}$$

Here, ω_p is the plasma frequency of some particle.

$$\nabla^2(\phi_0 + \delta\phi) = -\frac{1}{\epsilon_0}\sum e\int \mathbb{T}_S \bar{F} \delta(\boldsymbol{X}^* + \boldsymbol{\rho}_0 - \boldsymbol{r}) d^6\boldsymbol{Z}^*, \tag{32a}$$

$$\begin{aligned} &\nabla^2(\boldsymbol{A}_0 + \delta\boldsymbol{A}) \\ &= \frac{\omega_{pe}^2}{c^2}\delta\boldsymbol{A} - \mu_0 \sum e\int (\dot{\boldsymbol{X}}_0 + \dot{\xi}_0 \partial_\xi \boldsymbol{\rho}_0) \mathbb{T}_S \bar{F} \delta(\boldsymbol{X}^* + \boldsymbol{\rho}_0 - \boldsymbol{r}) d^6\boldsymbol{Z}^*. \end{aligned} \tag{32b}$$

Here, $\omega_{pi}^2/\omega_{pe}^2 = m_e/m_i \ll 1$ has been used. $\frac{\omega_{pe}^2}{c^2}\delta\boldsymbol{A}$ is the collisionless skin depth term. This term is first mentioned in Ref. 5 which only considers $\delta A_\parallel$ in the slab model.

The energy invariance can be written as

$$E_{total} = \int d^3x\left(\frac{\epsilon_0}{2}|\boldsymbol{E}_0+\delta\boldsymbol{E}|^2+\frac{1}{2\mu_0}|\boldsymbol{B}_0+\delta\boldsymbol{B}|^2\right) + \sum\int d^6\bar{Z}\bar{F}T_S^{-1}T_A^{-1}(H_0-e\phi_0), \tag{33a}$$

$$= \int d^3x\left(\frac{\epsilon_0}{2}|\boldsymbol{E}_0+\delta\boldsymbol{E}|^2+\frac{1}{2\mu_0}|\boldsymbol{B}_0+\delta\boldsymbol{B}|^2\right) + \sum\int d^6\bar{Z}\bar{F}(H_0-e\phi_0+\bar{H}_1-e\langle\delta\phi\rangle + \bar{H}_2+e\langle\{S_1,\delta\phi\}\rangle). \tag{33b}$$

The $\delta\boldsymbol{A}$ transform is easy to understand. It makes all the perturbations appear in the Hamiltonian part and keeps the structure of the Poisson bracket unchanged. The gyrokinetic theory based on the $\delta\boldsymbol{A}$ transform, which is similar to the kinetic[22] situation, agrees with the standard modern nonlinear gyrokinetic theory.[8] This will be demonstrated up to $O(\epsilon_\delta^2)$ in Appendix A. In addition, the second transform is determined by the perturbations only through the gauge functions S_n.

IV. SHORT TIME BEHAVIOR

A. I-transform method

The new phase-space transformation $\mathcal{T}_S^*$ called I transform[21–23] from the gyrocenter variables $\bar{Z}$ to new variables $\bar{Z}^*$ defined as $\bar{Z}^{*i}=\mathcal{T}_S^*\bar{Z}^i$ can be expanded in powers of the amplitude ordering parameter ϵ_δ up to $O(\epsilon_\delta^2)$, written as

$$\bar{Z}^{*i} = \bar{Z}^i + G_1^{*i} + G_2^{*i} + \frac{1}{2}G_1^{*j}\partial_j G_1^{*i}, \tag{34}$$

where $\boldsymbol{G}_1^*$ and $\boldsymbol{G}_2^*$ are the first-order and second-order generating vector fields, respectively. The transformed fundamental one-form is defined as

$$\hat{\bar{\Gamma}}^* \equiv \bar{\Gamma}^* - \bar{H}^*dt = \bar{\Gamma}_i^* d\bar{Z}^i - \bar{H}^*dt. \tag{35}$$

To make the transform equations of motion same as the unperturbed one, we set $\bar{\Gamma}_i^*=\Gamma_{0i}$ and $\bar{H}^*=H_0$, that is, $\hat{\bar{\Gamma}}^*\equiv\Gamma_0-H_0dt$. From the transform of the one-form, $\hat{\bar{\Gamma}}^* = \mathbb{T}_S^{*-1}\hat{\bar{\Gamma}}+dS^*$, we have

$$\bar{\Gamma}_{0i} = \Gamma_{0i}, \tag{36a}$$

$$0 = -G_n^{*j}\omega_{0ji} + \partial_i S_n^*(n=1,2), \tag{36b}$$

$$\bar{H}_0 = H_0, \tag{36c}$$

$$0 = \bar{H}_1 + \bar{H}_2 - G_1^{*i}\partial_i H_0, \tag{36d}$$

$$0 = -G_1^{*i}\partial_i\bar{H}_1 + \frac{1}{2}G_1^{*i}\partial_i(G_1^{*j}\partial_j H_0) - G_2^{*i}\partial_i H_0. \tag{36e}$$

Note that the second-order gyrocenter Hamiltonian $\bar{H}_2$ is placed in the first-order equation (36d).[23] From Eq. (36b), the nth-order generating vectors are

$$G_n^{*i} = \partial_j S_n^* J_0^{ji}(n=1,2). \tag{37}$$

Like $\boldsymbol{G}_n$, $\boldsymbol{G}_n^*$ are also the Hamiltonian flows and incompressible in the phase space, that is, $\frac{1}{\mathcal{J}}\partial_i(\mathcal{J}G_n^{*i})=0$. From Eqs. (36d) and (36e), the first-order and second-order gauge functions can be chosen as

$$\frac{d_0 S_1^*}{dt} = \delta\bar{H}, \tag{38a}$$

$$\frac{d_0 S_2^*}{dt} = -\frac{1}{2}\{S_1^*,\delta\bar{H}\}. \tag{38b}$$

With $\delta\bar{H}=\bar{H}-H_0=\bar{H}_1+\bar{H}_2$. The transformed equations of motion are

$$\frac{d\bar{Z}^{*i}}{dt} = J_0^{ij}\partial_j H_0, \tag{39}$$

which has the same form as the unperturbed ones. Then, the transformed gyrokinetic Vlasov equation is

$$\partial_t\bar{F}^* + \{\bar{F}^*, H_0\} = 0. \tag{40}$$

Note that Eq. (40) is valid in a short-time interval.[23] The transformed gyokinetic Vlasov equation is independent of the electromagnetic perturbations. By using the pull-back transform $\bar{F}=\mathbb{T}_S^*\bar{F}^*$ and the incompressibility of $\boldsymbol{G}_n^*$ in the phase space, we can find

$$\bar{F} = \bar{F}^* + \frac{1}{\mathcal{J}}\partial_i\left\{\mathcal{J}[(G_1^*+G_2^*)+\frac{1}{2}G_1^{*i}G_1^{*j}\partial_j]\bar{F}^*\right\}. \tag{41}$$

The gyrokinetic theory based on the I-transform method can be used to compute the guiding-center orbit and the guiding-center distribution function with the short-time approximation.[23] In Subsection IV B, we will discuss the short-time I-transform method and give the procedure of the computation of the guiding-center orbit and the guiding-center distribution function.

B. Application of the short-time I-transform method to the computation of the guiding-center orbit and the guiding-center distribution function

We choose a short time Δt, which satisfies

$$\epsilon_t = \dot{Z}_0^k\Delta t/L_k \ll 1, \tag{42}$$

with $Z^k=(X^i,\bar{v}_\parallel,t); L_k=\frac{H_n}{\partial_k H_n}$, that is, $(\omega,\boldsymbol{k}\cdot\boldsymbol{V},\dot{v}_{\parallel 0}/L_{v_\parallel})\times\Delta t\ll 1$. For the standard gyrokinetic ordering, we have $k_\parallel V_\parallel\sim\boldsymbol{k}_\perp\cdot\boldsymbol{V}_D$, then the condition becomes $\omega\Delta t,\omega_t\Delta t\ll 1$, with $\omega_t=k_\parallel V_\parallel$. Before pushing the gyrocenter of the particle every a short time, we choose

$$\bar{Z}(t-\Delta t) = \bar{Z}^*(t-\Delta t), \tag{43}$$

which means that two coordinates has the same phase-space point at $t-\Delta t$ and $\boldsymbol{G}_n(t-\Delta t)=0$. Thus, we can choose $S_n(t-\Delta t)=0$.

First, we make some general remarks on the computation of G_1^{*i}. S_1^* is kept up to $O(\epsilon_t^2)$, expressed as

$$S_1^*(\bar{Z}^*) = \int_{t-\Delta t}^t \delta\bar{H}[\bar{Z}^{*k}(\tau)]d\tau \simeq \int_{t-\Delta t}^t \delta\bar{H}[\bar{Z}^{*k}(t)+\dot{\bar{Z}}_0^{*k}(t)\times(\tau-t)]d\tau \simeq \delta\bar{H}\Delta t - \dot{\bar{Z}}_0^{*k}\partial_k\delta\bar{H}\frac{\Delta t^2}{2} = \delta\bar{H}\Delta t - \{\delta\bar{H},H_0\}_t\frac{\Delta t^2}{2}. \tag{44}$$

The subscript t denotes that the Poisson bracket is computed at the phase-space coordinates $\bar{Z}^*(t)$. Then we can find G_1^{*i} from Eq. (44)

$$G_1^{*i} = -J_0^{ij}\partial_j S_1^* \simeq -\{\bar{Z}^{*i}, \delta\bar{H}\}_t \Delta t + \{\bar{Z}^{*i}, \{\delta\bar{H}, H_0\}\}_t \frac{\Delta t^2}{2}. \quad (45)$$

Next, we will analyze G_2^{*i} by using the short-time approximation. S_2^* can be expressed as

$$S_2^* = -\frac{1}{2}\int_{t-\Delta t}^{t} \{S_1^*, \delta\bar{H}\} d\tau. \quad (46)$$

From Eq. (44), we have

$$\{S_1^*, \delta\bar{H}\}\Delta t = \{S_1^*, S_1^*\} + O(\epsilon_\delta^2 \epsilon_t^3) \sim O(\epsilon_\delta^2 \epsilon_t^3). \quad (47)$$

Then we can find

$$G_2^{*i} = -J_0^{ij}\partial_j S_2^* \sim O(\epsilon_\delta^2 \epsilon_t^3). \quad (48)$$

Thus, G_2^{*i} can be neglected when keeping up to $O(\epsilon_\delta^2 \epsilon_t^2)$.

The guiding-center orbit can be found from the gyrocenter orbit by the gyrocenter transform related to electromagnetic potential perturbations.[18] The procedure of the computation of the gyrocenter orbit based on the I-transform method can be expressed as

$$\bar{\boldsymbol{Z}}(t) = \mathcal{T}_S^{*-1}(t, t-\Delta t)[\bar{\boldsymbol{Z}}^*(t-\Delta t) + \Delta\boldsymbol{Z}_0], \quad (49)$$

which includes two steps: First, the equations of motion is integrated along unperturbed orbit during the short time Δt

$$\begin{aligned}\bar{Z}^{*i}(t) &= \bar{Z}^{*i}(t-\Delta t) + \Delta Z_0^i \\ &= \bar{Z}^{*i}(t-\Delta t) + \int_{t-\Delta t}^{t} \{\bar{Z}^{*i}, H_0\}_\tau d\tau. \end{aligned} \quad (50)$$

Second, we make the inverse I-transform ($\mathcal{T}_S^{*-1}(t, t-\Delta t)$)

$$\begin{aligned}\bar{Z}^i(t) &= \mathcal{T}_S^{*-1}(t, t-\Delta t)\bar{Z}^{*i}(t) \\ &= \bar{Z}^{*i} - G_1^{*i} + \frac{1}{2}G_1^{*j}\partial_j G_1^{*i}. \end{aligned} \quad (51)$$

Here, the inverse I-transform is kept up to $O(\epsilon_\delta^2 \epsilon_t^2)$, thus $\boldsymbol{G}_2^*$ is neglected due to Eq. (48). The new procedure of the gyrocenter orbit computation agrees with the conventional method, which integrate the gyrocenter equations of motion along the full orbit. This will be demonstrated in Appendix B. We will give two numerical examples about the guiding-center orbit computation by using the new procedure in Sec. V.

The guiding-center distribution function can be obtained from the gyrocenter distribution function by the pull-back transform (28) as is adopted in the standard modern gyrokinetic theory. The pull-back transform will give the adiabatic part of the guiding-center distribution function. Next, we discuss the computation of the gyrocenter distribution function by the I-transform method, which will give the nonadiabatic part. From Eq. (43), we have

$$\bar{F}(\bar{\boldsymbol{Z}} - \Delta\boldsymbol{Z}_0, t-\Delta t) = \bar{F}^*(\bar{\boldsymbol{Z}} - \Delta\boldsymbol{Z}_0, t-\Delta t), \quad (52)$$

which means the two distribution functions are same at $t - \Delta t$. By integrating Eq. (40) along the unperturbed orbit from $t - \Delta t$ to t, we can obtain

$$\bar{F}^*(\bar{\boldsymbol{Z}}, t) = \bar{F}^*(\bar{\boldsymbol{Z}} - \Delta\boldsymbol{Z}_0, t-\Delta t). \quad (53)$$

By using Eqs. (41), (52) and (53), we can find

$$\bar{F}(\bar{\boldsymbol{Z}}, t) = \bar{F}(\bar{\boldsymbol{Z}} - \Delta\boldsymbol{Z}_0, t-\Delta t) + \frac{1}{\mathcal{J}}\partial_i[\mathcal{J}(G_1^* + \frac{1}{2}G_1^{*i}G_1^{*j}\partial_j)\bar{F}]. \quad (54)$$

Here, $\boldsymbol{G}_2^*$ has been neglected. Equation (54) can be used to solve the gyrocenter distribution function by two steps.[23] The first step, which is related to the first term of the right-hand side of Eq. (54), is to compute the gyrocenter distribution function along the unperturbed orbit at $t - \Delta t$. The second step, which is shown in the second term, is to compute the effects of the electromagnetic perturbations. Equation (54) derived by using the I-transform method provides an alternative method for the nonlinear gyrokinetic simulation. The I-transform method, which splits the gyrocenter distribution function into the unperturbed part and the perturbation part, is similar to the δf-method. The character of the I-transform method is that all the perturbations are included in the first-order generating vector field $\boldsymbol{G}_1^*$. This character may be useful for theoretical analysis.

V. THE NUMERICAL CODE NLT AND ITS EXAMPLES

A. The code NLT

The numerical Lie-transform code NLT for computing the guiding-center orbit is developed, which is based on the short-time I-transform method presented above. Equations (37), (38a), (50), and (51) have been used in the code NLT to compute the gyrocenter orbit. The long-time computation of the gyrocenter orbit is composed of many short-time computations, that is, Eq. (49). In the code NLT, we choose the short time Δt as one time step. The procedure of the short-time computation has been discussed in Subsection V B.

To validate the code NLT, the code GYCAVA,[18] which has been developed earlier, is used as a benchmark in the following numerical examples. Both of them can be used to compute the guiding-center orbit in the tokamak configuration with a strong short-wavelength electromagnetic perturbation. The finite-Larmor-radius effect is included in the two codes. The code GYCAVA, based on the conventional method, pushes the phase-space point by integrating along the full gyrocenter orbit.

B. The numerical examples

In this subsection, we compare the new code NLT with the code GYCAVA. In both of the codes, the magnetic flux coordinates (ψ, θ, ζ) are used, with ψ being the poloidal magnetic flux, θ being the poloidal angle, and ζ being the toroidal angle. Two numerical examples are illustrated for validating the code NLT. The finite-Larmor-radius effect is not important in the two examples due to the character length of the perturbations used has the same order as the minor radius.

The first example is about the ion neoclassical polarization drift. The initial energy of the trapped ion is chosen as $E_{0i} = 10\,\text{keV}$. The unperturbed electric potential is not included, thus $H_0 = E_{0i} = 10\,\text{keV}$. The large time-varying electric potential perturbation is chosen as[34]

$$\delta\phi = \delta\phi_0 \psi \sin(\omega t). \tag{55}$$

Here, the mode frequency and amplitude are chosen as $\omega = 2.0 \times 10^4\text{Hz}, \delta\phi_0 = 2.0 \times 10^5\text{V/Wb}$, which makes the perturbation part of the Hamiltonian has the same order as the unperturbed one, that is, $\bar{H}_1 \sim H_0$. Note that $\psi \sim 0.1$ Wb for the model tokamak equilibrium used here.

The ion drifts radially, which is induced by the strong time-varying radial electric field, is shown in Fig. 1. It is clearly seen that both of the codes NLT and GYCAVA have the same orbit with the large electric potential perturbation.

The second example is about the gyrocenter motion in a large magnetic island perturbation. The perturbation of the poloidal magnetic flux is chosen as

$$\delta\psi = \delta\psi_{(m,n)} \cos(m\theta - n\zeta - \omega t) \tag{56}$$

to model a static magnetic island. The poloidal and toroidal numbers are chosen as $m = 2$, $n = 1$. The mode frequency and amplitude are chosen as $\omega = 0$, $\delta\psi_{(m,n)}/\psi_b = 10^{-3}$, with $\psi_b = 0.098$ Wb being the values of the equilibrium poloidal magnetic flux at the boundary of the tokamak. The initial particle energies of the electron and ion are set as $E_{0e} = 1\,\text{keV}$ and $E_{0i} = 100\,\text{keV}$. The Poincare section plots of the passing particles computed by the codes NLT and GYCAVA are shown in Fig. 2.

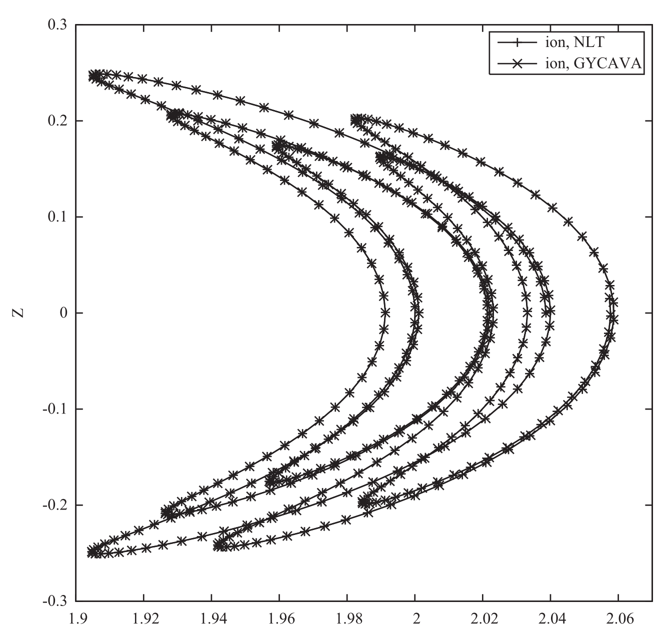

FIG. 1. Comparison of the neoclassical polarization drift of the trapped ion computed by NLT (labeled by plus and line) and by GYCAVA (labeled by cross and line), respectively.

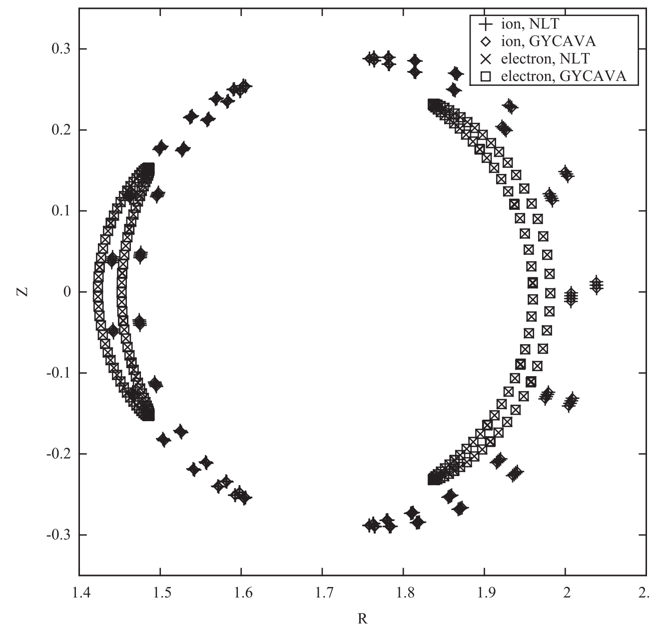

FIG. 2. Comparisons of the drift structure of the passing ion and the passing electron in a static magnetic field. The plus and the diamond symbols denote the Poincare plots of the passing ion computed by NLT and GYCAVA, respectively. The cross and the square symbols denote the Poincare plots of the passing electron computed by NLT and GYCAVA, respectively.

The Poincare plots of the passing particles have the drift island structure. The island structure of the passing electron is almost same as the magnetic island, while the island structure of the passing ion is shifted away from the magnetic field. From Fig. 2, it is clearly seen that both of the codes have the same Poincare plots for the electron and the ion.

From these two numerical examples, it can be seen that the guiding-center orbits computed by two different methods have the same results, which shows that the code NLT is valid for computing the guiding-center orbit in tokamaks.

VI. SUMMARY

The gyrokinetic theory in tokamak configuration based on the two-step transform has been presented. The two-step transform procedure includes the $\delta\boldsymbol{A}$ transform, which makes the effects of the magnetic potential perturbation appear in the Hamiltonian part, and the gyrokinetic Lie-transform, which transforms away the gyroangle-dependent part of the Hamiltonian to decouple the gyromotion from the gyrocenter motion. Note that the $\delta\boldsymbol{A}$ transform is generated by an incompressible flow in the phase space; the gyrokinetic Lie-transform presented in this paper is generated simply by a Hamiltonian flow. It has been proved that this gyrokinetic theory is equivalent with the conventional one. The two-step transform proposed in this paper is a generalization of Ref. 5, which considers $\delta A_{\parallel}$ in the slab model. The gyrokinetic Maxwell equations include the full electromagnetic potential perturbation $(\delta\phi, \delta\boldsymbol{A})$. The collisionless skin depth term due to the full magnetic potential perturbation $\delta\boldsymbol{A}$ has been recognized by using the $\delta\boldsymbol{A}$ transform.

The I-transform method is used to make the transformed equations of motion has the same form as the unperturbed ones. The application of the short-time I-transform method to the computation of the guiding-center orbit and the guiding-center distribution function has been presented. It is demonstrated analytically that the new method agrees with the conventional method in the orbit computation. A new code based on the short-time I-transform method is developed, which involves integrating only along the unperturbed orbit. Two numerical examples including the strong electromagnetic field perturbations have been illustrated to validate the new code by using a code which involves integrating along the full orbit in the conventional method. The unperturbed motion is well-understood so that the I-transform method provides an alternative method to compute the guiding-center orbit and the guiding-center distribution function. It can also be used and may bring about advantages in the nonlinear gyrokinetic simulation.

ACKNOWLEDGMENTS

This work was jointly supported by the National Natural Science Foundation of China under Grant No. 11175178, No. 11375196, the National ITER program of China under Contract No. 2014GB113000 and China Postdoctoral Science Foundation under Grant No. 2013M531509. We would like to acknowledge some numerical work by Zhikun Cai.

APPENDIX A: THE EQUIVALENCE OF THE GYROKINETIC THEORY BASED ON THE TWO-STEP TRANSFORM PROCEDURE AND THE STANDARD MODERN NONLINEAR GYROKINETIC THEORY

For the gyrokinetic theory based on the two-step transform procedure, the phase-space transform and the pull-back transform are

$$\begin{aligned}\bar{Z}^i = \mathcal{T}_S\mathcal{T}_A Z^i &= Z^i + \delta g_1^i + G_1^i + G_2^i \\ &+ \delta g_1^j\partial_j G_1^i + \frac{1}{2}G_1^j\partial_j G_1^i,\end{aligned} \tag{A1a}$$

$$\begin{aligned}F &= \mathbb{T}_A F^* = \mathbb{T}_A\mathbb{T}_S\bar{F} \\ &= \bar{F} + \Big(\delta g_1^i + G_1^i + G_2^i + \delta g_1^j\partial_j G_1^i \\ &\quad + \frac{1}{2}G_1^j\partial_j G_1^i + \frac{1}{2}\delta g_1^j\partial_j\delta g_1^i\Big)\partial_i\bar{F} \\ &\quad + \frac{1}{2}(\delta g_1^i\delta g_1^j + G_1^iG_1^j + 2\delta g_1^iG_1^j)\partial_{ij}\bar{F}.\end{aligned} \tag{A1b}$$

For the gyrokinetic theory based on the conventional[8] procedure, the phase-space transform and the pull-back transform are

$$\bar{Z}^{\dagger i} = \mathcal{T}_{\dagger}Z^i = Z^i + G_1^{\dagger i} + G_2^{\dagger i} + \frac{1}{2}G_1^{\dagger j}\partial_j G_1^{\dagger i}, \tag{A2a}$$

$$F^\dagger = \mathbb{T}_\dagger\bar{F}^\dagger = \bar{F}^\dagger + (G_1^{\dagger i} + G_2^{\dagger i} + G_1^{\dagger j}\partial_j G_1^{\dagger i})\partial_i\bar{F}^\dagger + \frac{1}{2}G_1^{\dagger i}G_1^{\dagger j}\partial_{ij}\bar{F}^\dagger. \tag{A2b}$$

Here, the conventional first-order generating vector $\boldsymbol{G}_1^\dagger$ can be expressed as

$$G_1^{\dagger i} = \delta g_1^i + G_1^i. \tag{A3}$$

The conventional second-order generating vector $\boldsymbol{G}_2^\dagger$ can be expressed as

$$G_2^{\dagger i} = \{S_2^\dagger, Z^i\} - \frac{1}{2}G_1^{\dagger k}\omega_{1kj}J_0^{ji}. \tag{A4}$$

The conventional gauge function $S_2^\dagger$ and the new gauge function S_2 have the following relations:

$$S_2^\dagger = S_2 - \frac{1}{2}\{S_1, Z^j\}\Gamma_{1j}. \tag{A5}$$

From Eqs. (A4), (A5), and (21), the conventional second-order generating vector $\boldsymbol{G}_2^\dagger$ can be rewritten as

$$G_2^{\dagger i} = G_2^i - \left\{\frac{1}{2}\{S_1, Z^j\}\Gamma_{1j}, Z^i\right\} - \frac{1}{2}G_1^k\omega_{1kj}J_0^{ji}. \tag{A6}$$

Then, we can obtain

$$G_2^{\dagger i} + \frac{1}{2}G_1^{\dagger j}\partial_j G_1^{\dagger i} = G_2^i + \delta g^j\partial_j G_1^i + \frac{1}{2}G_1^j\partial_j G_1^i + \frac{1}{2}\delta g_1^j\partial_j\delta g_1^i. \tag{A7}$$

From Eqs. (A1a), (A2a), (A3), and (A7), we can see that

$$\bar{Z}^i = \bar{Z}^{\dagger i}. \tag{A8}$$

From Eq. (A8) and the property of the scaler invariance $F(Z) = \bar{F}(\bar{Z}) = \bar{F}^\dagger(\bar{Z}^\dagger)$, we have

$$\bar{F} = \bar{F}^\dagger. \tag{A9}$$

Using Eqs. (A9), (A1b), (A2b), (A3), and (A7), we can see that

$$F = F^\dagger. \tag{A10}$$

Equations (A8), (A9), and (A10) mean that the gyrokinetic theory based on the two-step transform procedure agrees with the standard modern nonlinear gyrokinetic theory.[8]

APPENDIX B: THE EQUIVALENCE OF THE SHORT-TIME I-TRANSFORM METHOD AND CONVENTIONAL METHOD FOR COMPUTING GYROCENTER ORBIT

According to the conventional[18] method, the change of the phase-space position $\Delta\bar{\boldsymbol{Z}}^\dagger$ during a short time Δt is kept up to $O(\epsilon_\delta^2\epsilon_t^2)$, written as

$$\begin{aligned}\Delta\bar{Z}^{\dagger i} &= \int_{t-\Delta t}^{t}\{\bar{Z}^i, \bar{H}\}_\tau d\tau \\ &\simeq \{\bar{Z}^i, \bar{H}\}_{t-\Delta t}\Delta t + \{\{\bar{Z}^i, \bar{H}\}, \bar{H}\}_{t-\Delta t}\frac{\Delta t^2}{2}.\end{aligned} \tag{B1}$$

According to the short-time I-transform method, from the procedure of gyrocenter orbit computation, the change of the phase-space position $\Delta\bar{\boldsymbol{Z}}$ during a short time Δt is kept up to $O(\epsilon_\delta^2\epsilon_t^2)$, written as

$$\Delta \bar{Z}^i = \Delta \bar{Z}^{*i} - G_1^{*i} + \frac{1}{2} G_1^{*j} \partial_j G_1^{*i}. \tag{B2}$$

From Eq. (45), the first-order generating vector can be rewritten as

$$\begin{aligned} G_1^{*i} &\simeq -(\{\bar{Z}^{*i}, \delta\bar{H}\}_{t-\Delta t} + \{\{\bar{Z}^{*i}, \delta\bar{H}\}, H_0\}_{t-\Delta t}\Delta t)\Delta t \\ &\quad -(\{\bar{H}_0, \{\bar{Z}^{*i}, \delta\bar{H}\}\}_{t-\Delta t} - \{\delta\bar{H}, \{\bar{Z}^{*i}, H_0\}\}_{t-\Delta t})\frac{\Delta t^2}{2} \\ &= -\{\bar{Z}^{*i}, \delta\bar{H}\}_{t-\Delta t}\Delta t \\ &\quad -(\{\{\bar{Z}^{*i}, \delta\bar{H}\}, H_0\}_{t-\Delta t} + \{\{\bar{Z}^{*i}, H_0\}, \delta\bar{H}\}_{t-\Delta t})\frac{\Delta t^2}{2}. \end{aligned} \tag{B3}$$

Using Eq. (B3) and keep up to $O(\epsilon_\delta^2 \epsilon_t^2)$, we have

$$\begin{aligned} \frac{1}{2} G_1^{*j} \partial_j G_1^{*i} &= \frac{1}{2}\{S_1^*, \{S_1^*, \bar{Z}^{*i}\}\}_{t-\Delta t} \\ &\simeq \frac{1}{2}\{\{\bar{Z}^{*i}, \delta\bar{H}\}, \delta\bar{H}\}_{t-\Delta t}\frac{\Delta t^2}{2}. \end{aligned} \tag{B4}$$

From Eq. (50), the change of the phase-space position $\Delta \bar{\mathbf{Z}}^*$ during a short time Δt can be computed by integrating along the unperturbed orbit

$$\begin{aligned} \Delta \bar{Z}^{*i} &= \int_{t-\Delta t}^{t} \{\bar{Z}^{*i}, H_0\}_\tau d\tau \\ &\simeq \{\bar{Z}^{*i}, \bar{H}_0\}_{t-\Delta t}\Delta t + \{\{\bar{Z}^{*i}, H_0\}, H_0\}_{t-\Delta t}\frac{\Delta t^2}{2}. \end{aligned} \tag{B5}$$

Using Eqs. (B1)–(B5), we have

$$\Delta \bar{Z}^i = \Delta \bar{Z}^{\dagger i}. \tag{B6}$$

From Eq. (B6), we can see that the change of the phase-space position during a short time Δt computed by the new method is same as the one computed by the conventional method up to $O(\epsilon_\delta^2 \epsilon_t^2)$. It is demonstrated that two methods for computing the gyrocenter orbit in tokamaks are equivalent.

[1] T. M. Antonsen, Jr. and B. Lane, Phys. Fluids **23**, 1205 (1980).
[2] P. J. Catto, W. M. Tang, and D. E. Baldwin, Plasma Phys. **23**, 639 (1981).
[3] E. A. Frieman and L. Chen, Phys. Fluids **25**, 502 (1982).
[4] D. H. E. Dubin, J. A. Krommes, C. Oberman, and W. W. Lee, Phys. Fluids **26**, 3524 (1983).
[5] T. S. Hahm, W. W. Lee, and A. J. Brizard, Phys. Fluids **31**, 1940 (1988).
[6] T. S. Hahm, Phys. Fluids **31**, 2670 (1988).
[7] A. J. Brizard, J. Plasma Phys. **41**, 541 (1989).
[8] A. J. Brizard and T. S. Hahm, Rev. Mod. Phys. **79**, 421 (2007).
[9] W. W. Lee, Phys. Fluids **26**, 556 (1983).
[10] S. E. Parker and W. W. Lee, Phys. Fluids B **5**, 77 (1993).
[11] M. Kotschenreuther, G. Rewoldt, and W. M. Tang, Comput. Phys. Commun. **88**, 128 (1995).
[12] Z. Lin, T. S. Hahm, W. W. Lee, W. M. Tang, and R. B. White, Science **281**, 1835 (1998).
[13] F. Jenko, W. Dorland, M. Kotschenreuther, and B. N. Rogers, Phys. Plasmas **7**, 1904 (2000).
[14] J. Candy and R. E. Waltz, J. Comput. Phys. **186**, 545 (2003).
[15] X. Q. Xu, Z. Xiong, M. R. Dorr, J. A. Hittinger, K. Bodi, J. Candy, B. I. Cohen, R. H. Cohen, P. Colella, G. D. Kerbel, S. Krasheninnikov, W. M. Nevins, H. Qin, T. D. Rognlien, P. B. Snyder, and M. V. Umansky, Nucl. Fusion **47**, 809 (2007).
[16] X. Garbet, Y. Idomura, L. Villard, and T. H. Watanabe, Nucl. Fusion **50**, 043002 (2010).
[17] S. Wang, Phys. Rev. E **64**, 056404 (2001).
[18] Y. Xu, X. Xiao, and S. Wang, Phys. Plasmas **18**, 042505 (2011).
[19] Y. Xu and S. Wang, Plasma Phys. Controlled Fusion **55**, 015009 (2013).
[20] Y. Xu and S. Wang, Phys. Plasmas **20**, 012515 (2013).
[21] S. Wang, Phys. Plasmas **19**, 062504 (2012).
[22] S. Wang, Phys. Rev. E **87**, 063103 (2013).
[23] S. Wang, Phys. Plasmas **20**, 082312 (2013).
[24] R. G. Littlejohn, J. Math. Phys. **20**, 2445 (1979).
[25] J. R. Cary, Phys. Rep. **79**, 129 (1981).
[26] R. G. Littlejohn, J. Math. Phys. **23**, 742 (1982).
[27] J. R. Cary and R. G. Littlejohn, Ann. Phys. (N.Y.) **151**, 1 (1983).
[28] J. R. Cary and A. N. Kaufman, Phys. Fluids **24**, 1238 (1981).
[29] R. G. Littlejohn, J. Plasma Phys. **29**, 111 (1983).
[30] R. G. Littlejohn, Phys. Fluids **24**, 1730 (1981).
[31] A. J. Brizard, Phys. Plasmas **2**, 459 (1995).
[32] T. S. Hahm, Phys. Plasmas **3**, 4658 (1996).
[33] H. Goldstein, C. Poole, and J. Safko, *Classical Mechanics* (Addison-Wesley, 2002), Chap. 9, p. 392.
[34] C. Pan, X. Xiao, and S. Wang, Phys. Plasmas **18**, 032514 (2011).