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To cite this article: Jing-Rong Wang *et al* 2016 *New J. Phys.* **18** 073023

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OPEN ACCESS

RECEIVED

7 February 2016

REVISED

1 June 2016

ACCEPTED FOR PUBLICATION

27 June 2016

PUBLISHED

12 July 2016

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At the nematic quantum critical point that exists in the $d_{x^2-y^2}$ -wave superconducting dome of cuprates, the massless nodal fermions interact strongly with the quantum critical fluctuation of nematic order. We study this problem by means of the renormalization group approach and show that, the fermion damping rate $|\text{Im} \Sigma^R(\omega)|$ vanishes more rapidly than the energy ω and the quasiparticle residue $Z_f \rightarrow 0$ in the limit $\omega \rightarrow 0$. The nodal fermions thus constitute an unconventional non-Fermi liquid that represents an even weaker violation of Fermi liquid theory than a marginal Fermi liquid. We also investigate the interplay of quantum nematic critical fluctuation and gauge-potential-like disorder, and find that the effective disorder strength flows to the strong coupling regime at low energies. Therefore, even an arbitrarily weak disorder can drive the system to become a disorder controlled diffusive state. Based on these theoretical results, we are able to understand a number of interesting experimental facts observed in cuprate superconductors.

1. Introduction

A large amount of experimental and theoretical studies have been devoted to studying the unusual properties of high temperature cuprate superconductors in the past thirty years [1–9]. Although some consensus have been reached, many fundamental problems are still in debate, including the microscopic pairing mechanism [1, 2, 7–9], the origin of pseudogap [2, 6], and the description of non-Fermi liquid behaviors of the normal state [1–3]. In the past decade, there have been accumulating experimental evidences for the existence of a strong anisotropy in many of the physical properties of $\text{YBa}_2\text{Cu}_3\text{O}_{6+\delta}$ (YBCO) [10–12] and $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$ (BSCCO) [13, 14]. Such an anisotropy is widely believed to be driven by the formation of a novel electronic nematic order [15–19], which spontaneously breaks the C_4 symmetry down to a C_2 symmetry. In case the nematic transition line goes across the superconducting transition line and penetrates into the superconducting dome, there exists a zero-temperature nematic quantum critical point (QCP). The nematic quantum phase transition and the associated quantum critical behaviors have been investigated extensively in recent years [20–31].

From a theoretical perspective, there are two widely studied scenarios to induce an electronic nematic order. First, the nematic order can be generated by melting a stripe order that spontaneously breaks both translational and rotational symmetry [15–19]. The other way is related to Pomeranchuk instability which refers to the deformation of the shape of the Fermi surface of a metal due to Coulomb interaction [15–19, 32–36]. In the simplest case, Pomeranchuk instability occurs when the circular Fermi surface of a two-dimensional metal becomes ellipse-like via quadrupolar distortion. [15–19]. The Hubbard model defined on square lattices [35, 37–44] provides a pertinent platform to investigate the electronic nematic order. Halboth and Metzner [35] studied a two-dimensional Hubbard model using functional renormalization group (RG) method, and revealed Pomeranchuk instability and nematic order. More recently, the square lattice Hubbard model is studied by variational cluster approximation [45, 46] and found to display a local nematic phase under certain

circumstances, which might be applied to understand the intra-unit-cell electronic nematicity observed in the scanning tunneling spectroscopy (STS) measurements by Lawler *et al* [13].

Another context of studying nematic order is provided by the superconducting dome of cuprate superconductors. For a pure $d_{x^2-y^2}$ -wave superconductor, the discrete C_4 symmetry is preserved. However, when superconductivity coexists with a nematic order, the gap nodes are shifted from their original positions and the C_4 symmetry is broken down to C_2 [20, 22–24]. Such an anisotropic superconducting state is physically equivalent to a $d_{x^2-y^2} + s$ -wave superconducting state [20, 22]. At the nematic QCP, the massless fermions excited from the $d_{x^2-y^2}$ -wave gap nodes couple strongly to the quantum critical fluctuation of nematic order parameter, which can be effectively described by a $(2 + 1)$ -dimensional field theory [20–31]. This model was first analyzed by Vojta *et al* [20–22], who made an ϵ -expansion and argued that the nematic phase transition is turned to first order. Kim *et al* [23] later tackled the same model by means of $1/N$ -expansion, where N is a large fermion flavor, and concluded that the transition remains continuous. Huh and Sachdev [24] performed a renormalization group analysis, and showed that the fermion velocity ratio $v_{\Delta}/v_F \rightarrow 0$ in the lowest energy limit, where v_F is the Fermi velocity of nodal fermions and v_{Δ} the gap velocity [20–31]. The unusual velocity renormalization leads to significant changes of a number of quantities, including density of states (DOS) [25], specific heat [25], low- T thermal conductivity [26], superfluid density [28], and London penetration depth [31].

In this article, we revisit the issue of unusual physical properties caused by the strong interaction between massless nodal fermions and critical nematic fluctuation. We will apply the powerful RG approach to calculate the fermion damping rate $|\text{Im} \Sigma^R(\omega)|$, where $\text{Im} \Sigma^R(\omega)$ is the imaginary part of retarded fermion self-energy, and the corresponding quasiparticle residue Z_f . Kim *et al* [23] have previously computed the fermion self-energy and spectral function. Interestingly, it will become clear below that the RG analysis give rise to different results once the singular renormalization of fermion velocities is taken into account. In particular, we will show that the fermion damping rate vanishes upon approaching the Fermi level more rapidly than the energy ω , namely $\lim_{\omega \rightarrow 0} |\text{Im} \Sigma^R(\omega)|/\omega \rightarrow 0$. According to the conventional notion of quantum many-particle physics, one would expect the system to behave as a normal Fermi liquid. However, by analyzing the RG results, we find that the quasiparticle residue vanishes, i.e., $Z_f \rightarrow 0$, in the limit $\omega \rightarrow 0$. Therefore, the system is actually a non-Fermi liquid that represents an even weaker violation of Fermi liquid theory comparing to a marginal Fermi liquid (MFL). To the best of our knowledge, this type of unconventional non-Fermi liquid behavior has not been reported previously.

In realistic materials, there are always certain amount of disorders, which may play an important role. As demonstrated earlier by Nersisyan *et al* [47, 48], the most important disorder in cuprates behaves like a randomly distributed gauge potential. Thus, we will mainly study the influence of random gauge potential on the physical properties of nodal fermions and also the stability of nematic QCP. In the absence of nematic order, the coupling between nodal fermions and random gauge potential has attracted considerable interest [47–49]. Here, we consider the case in which nodal fermions couple to both the nematic order and random gauge potential, and then study the interplay of these two interactions by means of RG method. We find that the effective strength of gauge potential disorders tends to diverge at the lowest energy. This behavior signals the emergence of a finite zero-energy DOS $\rho(0)$ and the happening of quantum phase transition from an unconventional non-Fermi liquid to a disorder dominated diffusive state. The nodal fermions acquire a finite scattering rate γ , which in turn affects the thermodynamic and spectral behaviors of nodal fermions.

The RG results for the self-energy of nodal fermions can be used to understand a number of experimental facts observed in cuprate superconductors. We will show that the RG results are qualitatively consistent with some recent measurements of specific heat, fermion damping rate, and temperature dependence of fermion velocities.

The rest of the paper will be organized as follows. We first present the effective low-energy field theory for the interaction between nematic order and nodal fermions in section 2. The random gauge potential is also introduced in this section. In section 3, we make detailed theoretical analysis for the self-consistent RG equations of the fermion velocities in the clean case. Based on the RG solutions, we proceed to compute the fermion damping rate, quasiparticle residue, and other physical quantities. We will show that the quantum critical fluctuation of nematic order leads to unconventional non-Fermi liquid behaviors of nodal fermions. We consider the influence of random gauge potential in section 4 and find that the effective disorder strength flows to the strong coupling regime at low energies. Therefore, even weak disorders play a significant role and drive the system to enter into a disorder controlled diffusive state. In section 5, we discuss the possible application of the RG results to some experimental findings of cuprates. In section 6, we briefly summarize our main results.

2. Effective field theory

We start from an effective action $S = S_\Psi + S_\phi + S_{\Psi\phi}$. The free action for the nodal fermions is [20–24]

$$S_\Psi = \int \frac{d\omega d^2\mathbf{k}}{(2\pi)^3} \Psi_{1a}^\dagger (-i\omega + v_F k_x \tau^z + v_\Delta k_y \tau^x) \Psi_{1a} + \int \frac{d\omega d^2\mathbf{k}}{(2\pi)^3} \Psi_{2a}^\dagger (-i\omega + v_F k_y \tau^z + v_\Delta k_x \tau^x) \Psi_{2a}, \quad (1)$$

where $\tau^{(x,y,z)}$ denote the Pauli matrices. The two component Nambu spinors are defined by $\Psi_{1a} = (f_{1a}, \epsilon_{ab} f_{3b}^\dagger)^T$ and $\Psi_{2a} = (f_{2a}, \epsilon_{ab} f_{4b}^\dagger)^T$ where $\epsilon_{ab} = -\epsilon_{ba}$, f_{1a} , f_{2a} , f_{3a} and f_{4a} represent fermions excited from the nodal points (K, K) , $(-K, K)$, $(-K, -K)$, and $(K, -K)$ respectively [20–24]. The repeated spin index a is summed from 1 to N , where N is the number of fermion spin components with the physical value being 2. The action S_ϕ describes the nematic order parameter, which is expanded in real space as

$$S_\phi = \int d\tau d^2\mathbf{x} \left[\frac{1}{2} (\partial_\tau \phi)^2 + \frac{c^2}{2} (\nabla \phi)^2 + \frac{r}{2} \phi^2 + \frac{u_0}{24} \phi^4 \right], \quad (2)$$

where τ is imaginary time and c velocity of field ϕ . It is convenient to choose $c = 1$. Mass parameter r tunes a quantum phase transition from $d_{x^2-y^2}$ -wave superconducting state to a state where the superconducting and nematic orders coexist, with $r = 0$ defining the QCP. Moreover, u_0 is the quartic self-interaction strength. The nematic order couples to fermions via a Yukawa-coupling term

$$S_{\Psi\phi} = \lambda_0 \int d\tau d^2\mathbf{x} \phi (\Psi_{1a}^\dagger \tau^x \Psi_{1a} + \Psi_{2a}^\dagger \tau^x \Psi_{2a}), \quad (3)$$

where λ_0 is the coupling constant. In addition to the nematic order, the nodal fermions also couple to gauge-potential type disorders, which is described by

$$S_{\text{dis}} = v_\Gamma \int d^2\mathbf{x} A(\mathbf{x}) (\Psi_{1a}^\dagger \Gamma \Psi_{1a} + \Psi_{2a}^\dagger \Gamma \Psi_{2a}). \quad (4)$$

with $\Gamma = (\tau^z, \tau^x)$ and $v_\Gamma = (v_{\Gamma 1}, v_{\Gamma 2})$. Here, the random gauge potential $A(\mathbf{x})$ is assumed to be a Gaussian white noise, defined by

$$\langle A(\mathbf{x}) \rangle = 0, \quad \langle A(\mathbf{x}) A(\mathbf{x}') \rangle = g \delta^2(\mathbf{x} - \mathbf{x}'), \quad (5)$$

where g is the impurity concentration and v_Γ measures the strength of a single impurity.

We will follow Huh and Sachdev [24] and perform RG analysis by employing $1/N$ -expansion. The inverse of free propagator of ϕ behaves as $q^2 + r$. After including the polarization, there will be an additional linear-in- q term. The q -term dominates at small q over q^2 -term, so the q^2 -term can be neglected. Near the nematic QCP, we keep only the mass term and rescale $\phi \rightarrow \phi/\lambda_0$ and $r \rightarrow N_f r \lambda_0^2$, leading to

$$S = S_\Psi + \int d\tau d^2\mathbf{x} \left\{ \frac{N_f r}{2} \phi^2 + \phi [\Psi_{1a}^\dagger \tau^x \Psi_{1a} + \Psi_{2a}^\dagger \tau^x \Psi_{2a}] \right\}. \quad (6)$$

The free propagators for fermions Ψ_{1a} and Ψ_{2a} are written as

$$G_{1a}^0(\omega, \mathbf{k}) = \frac{1}{-i\omega + v_F k_x \tau^z + v_\Delta k_y \tau^x}, \quad (7)$$

$$G_{2a}^0(\omega, \mathbf{k}) = \frac{1}{-i\omega + v_F k_y \tau^z + v_\Delta k_x \tau^x}, \quad (8)$$

respectively. At the nematic QCP, $r = 0$, the propagator for the nematic order field ϕ is

$$D(\Omega, \mathbf{q}) = \frac{1}{\Pi(\Omega, \mathbf{q})}, \quad (9)$$

where the polarization $\Pi(\Omega, \mathbf{q})$, to the leading order of $1/N$ -expansion, is given by

$$\begin{aligned} \Pi(\Omega, \mathbf{q}) &= \sum_{i=1,2} \sum_{a=1}^N \int \frac{d\omega}{2\pi} \frac{d^2\mathbf{k}}{(2\pi)^2} \text{Tr} [\tau^x G_{ia}^0(\mathbf{k}, \omega) \tau^x G_{ia}^0(\mathbf{k} + \mathbf{q}, \omega + \Omega)] \\ &= \frac{N}{16v_F v_\Delta} \frac{\Omega^2 + v_F^2 q_x^2}{\sqrt{\Omega^2 + v_F^2 q_x^2 + v_\Delta^2 q_y^2}} + (q_x \leftrightarrow q_y). \end{aligned} \quad (10)$$

After carrying out RG calculations [24, 27], we find that all the physical parameters flow with a varying length scale l as follows

$$\frac{dv_F}{dl} = (C_1 - C_2 - C_g)v_F, \quad (11)$$

$$\frac{dv_\Delta}{dl} = (C_1 - C_3 - C_g)v_\Delta, \quad (12)$$

$$\frac{d(v_\Delta/v_F)}{dl} = (C_2 - C_3)\frac{v_\Delta}{v_F}, \quad (13)$$

$$\frac{dv_{\Gamma_1}}{dl} = (C_1 - C_2 - C_g)v_{\Gamma_1}, \quad (14)$$

$$\frac{dv_{\Gamma_2}}{dl} = (C_1 - C_3 - C_g)v_{\Gamma_2}, \quad (15)$$

where we have introduced a quantity

$$C_g = \frac{v_{\Gamma_1}^2 + v_{\Gamma_2}^2}{2\pi v_F v_\Delta} g. \quad (16)$$

The expressions of $C_{1,2,3}$ can be found in the A. We emphasize that the impact of disorders is characterized by the quantity C_g , rather than v_{Γ_i} with $i = 1, 2$. It will be shown below that C_g flows to strong coupling at large l , leading to remarkable physical properties.

3. Unconventional non-Fermi liquid behaviors of nodal fermions

It is well established that conventional metals can be described by the standard Fermi liquid theory, which states that the fermionic excitations of a normal Fermi liquid must have a sufficiently long lifetime and exhibit a sharp quasiparticle peak in their spectral peak despite the existence of Coulomb interaction [50]. The conventional notion is that the fermionic quasiparticles constitute a normal Fermi liquid if their zero- T damping rate $|\text{Im } \Sigma^R(\omega)|$ vanishes more rapidly than ω in the limit $\omega \rightarrow 0$. This criterion can be mathematically expressed as

$$\lim_{\omega \rightarrow 0} \frac{\text{Im } \Sigma^R(\omega)}{\omega} \rightarrow 0. \quad (17)$$

Another criterion to identify normal Fermi liquid is to define an important quantity: the quasiparticle residue, also called renormalization factor, Z_f . The residue Z_f is usually calculated through the definition

$$Z_f(\omega) = \frac{1}{\left| 1 - \frac{\partial}{\partial \omega} \text{Re } \Sigma^R(\omega) \right|}, \quad (18)$$

where the real part of retarded fermion self-energy $\text{Re } \Sigma^R(\omega)$ is related to $\text{Im } \Sigma^R(\omega)$ via the Kramers-Kronig (K.-K.) relationship. The residue Z_f is finite in a normal Fermi liquid, but vanishes in a non-Fermi liquid. Generically, the fermion damping rate in an interacting fermion system can be formally written as

$$\text{Im } \Sigma^R(\omega) = C_F |\omega|^x, \quad (19)$$

where C_F is a constant. With the help of K.-K. relation, the corresponding real part of the fermion self-energy, in the low energy limit, is given by

$$\text{Re } \Sigma^R(\omega) = \begin{cases} C_F \text{sgn}(\omega) |\omega|^x I(x), & \text{if } 0 < x < 1, \\ C_F \frac{2}{\pi} \omega \ln\left(\frac{\omega_0}{|\omega|}\right), & \text{if } x = 1, \\ C_F \frac{2}{\pi} \frac{|\omega_0|^{x-1}}{x-1} \omega, & \text{if } x > 1, \end{cases} \quad (20)$$

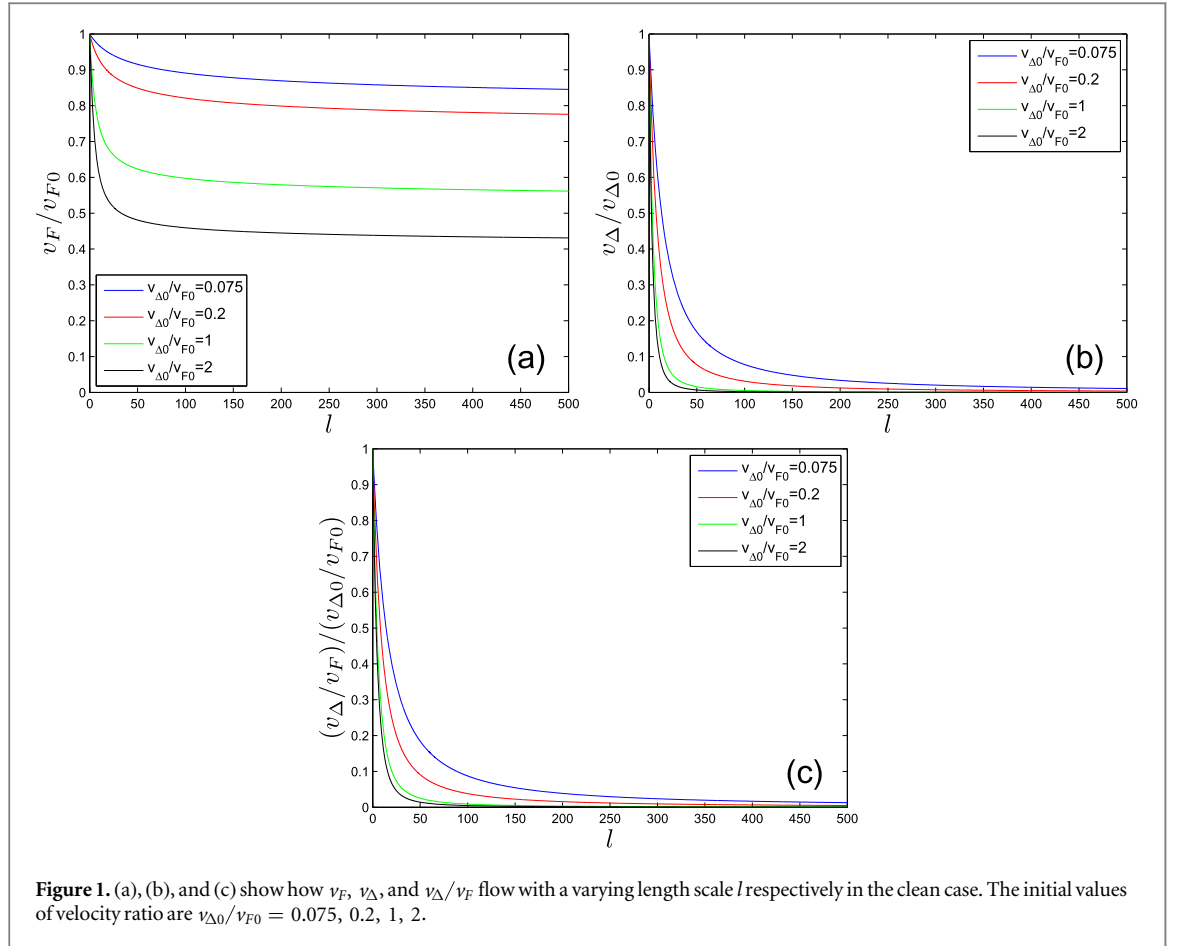
where ω_0 is a cutoff, and $I(x)$ is a function that depends only on x .

It can be easily checked that $Z_f = 0$ for $0 < x \leq 1$ and $Z_f \neq 0$ for $x > 1$. Therefore, the above two criteria are actually equivalent because Z_f automatically takes a finite value whenever equation (17) is fulfilled.

However, in the present nodal fermion system, the above two criteria are no longer equivalent. We will show by explicit calculations in this section that the damping rate of nodal fermions vanishes more rapidly than the energy, but the residue Z_f vanishes, i.e., $Z_f \rightarrow 0$.

3.1. Fermion damping rate and quasiparticle residue

Now we calculate the fermion damping rate and the residue Z_f utilizing the solutions of the RG equations (11)–(15). The unusual renormalization of fermion velocities need to be taken into account in an appropriate manner. We only present the results obtained in the clean limit $C_g = 0$ in this section, and include the effect of random gauge potential in the next section.



After solving RG equations (11)–(13) self-consistently, we show the l -dependence of fermion velocities v_F , v_{Δ} and ratio v_{Δ}/v_F obtained at different initial values of ratio $v_{\Delta 0}/v_{F0}$ in figures 1(a), (b) and (c) respectively. All the quantities v_F , v_{Δ} , and v_{Δ}/v_F decrease with growing l and flow eventually to zero as $l \rightarrow +\infty$, but apparently v_F decreases much more slowly than v_{Δ} . If we use the ratio v_{Δ}/v_F to characterize the velocity anisotropy, it is clear that the nematic order drives an extreme velocity anisotropy $v_{\Delta}/v_F \rightarrow 0$. For later use, it is helpful to extract an approximate analytical expressions for the velocities. Considering the leading and sub-leading terms, the solution for v_{Δ}/v_F is given by

$$\frac{v_{\Delta}}{v_F} \sim \frac{\pi^2 N}{8} \frac{1}{l \ln(c_s l)} = \frac{\pi^2 N}{8} \frac{1}{l [\ln(l) + \ln(c_s)]}, \quad (21)$$

with $c_s \approx 0.3809/N$, which is consistent with the expression in [24]. In the long wavelength limit, $l \rightarrow \infty$, $\ln(c_s)/\ln(l) \rightarrow 0$, so the sub-leading term can be ignored. Retaining only the leading term, the asymptotic behavior of velocity ratio can be well approximated by

$$\frac{v_{\Delta}}{v_F} \sim \frac{\pi^2 N}{8} \frac{1}{l \ln(l)}. \quad (22)$$

Substituting the asymptotic form of v_{Δ}/v_F into equation (11), we obtain

$$\frac{dv_F}{dl} \sim -\frac{\pi^2 c_1}{8} \frac{v_F}{l \ln(l)} \quad (23)$$

with $c_1 \approx 0.078$ for large l . Solving this equation, we express the renormalized v_F as

$$v_F(l) \sim \frac{1}{(\ln l)^{c_2}} \quad (24)$$

with $c_2 = \frac{\pi^2 c_1}{8}$. It can also be found that v_{Δ} at large l behaves approximately as

$$v_{\Delta}(l) \sim \frac{\pi^2 N}{8} \frac{1}{l [\ln(l)]^{1+c_2}}. \quad (25)$$

According to equations (24) and (25), both v_F and v_{Δ} vanish as $l \rightarrow +\infty$.

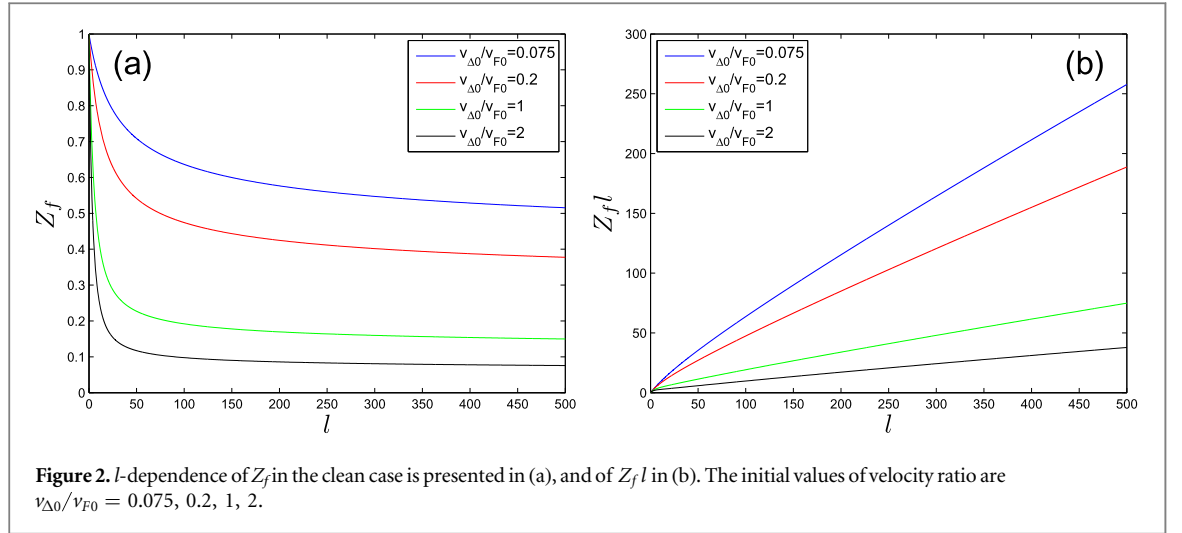


Figure 2. l -dependence of Z_f in the clean case is presented in (a), and of $Z_f l$ in (b). The initial values of velocity ratio are $v_{\Delta 0}/v_{F0} = 0.075, 0.2, 1, 2$.

To examine the impact of singular fermion velocity renormalization and extreme velocity anisotropy on the properties of nodal fermions, we will compute a number of physical quantities. Coming first is the quasiparticle residue Z_f . Apart from the widely used definition given by equation (18), the residue Z_f can also be calculated within the RG framework. The interaction induced renormalization of fermion field Ψ is encoded in the residue Z_f which exhibits the following l -dependence

$$Z_f = e^{\int_0^l (C_1 - C_g) dl'} \quad (26)$$

or alternatively

$$\frac{dZ_f}{dl} = (C_1 - C_g)Z_f. \quad (27)$$

The l -dependence of Z_f can be easily obtained from the above equation, and is presented in figure 2(a). We find that Z_f flows to zero in the limit $l \rightarrow +\infty$, which indicates the breakdown of normal Fermi liquid and the absence of well-defined Landau quasiparticles. However, Z_f decreases very slowly with growing l , thus the deviation of the system from a normal Fermi liquid ought to be quite weak. To see this point, we plot the l -dependence of $Z_f l$ in figure 2(b), and find that

$$\lim_{l \rightarrow +\infty} Z_f l \rightarrow +\infty. \quad (28)$$

Therefore, the residue Z_f obtained at nematic QCP vanishes with growing l more slowly than that of a MFL [51], where $Z_f \sim 1/l$. According to equation (18), one can speculate that

$$\lim_{\omega \rightarrow 0} \frac{\text{Re} \Sigma_{\text{nem}}^R(\omega)}{\omega \ln(\omega_0/\omega)} \rightarrow 0, \quad (29)$$

since a large length scale l corresponds to a low energy ω . This speculation in turn implies that the imaginary part of the retarded fermion self-energy $\text{Im} \Sigma_{\text{nem}}^R(\omega)$ should display the following low-energy behavior,

$$\lim_{\omega \rightarrow 0} \frac{\text{Im} \Sigma_{\text{nem}}^R(\omega)}{\omega} \rightarrow 0. \quad (30)$$

To make the above discussions more quantitative, we are now going to make a detailed analysis of the asymptotic behavior of Z_f . At large running scale l , the equation of Z_f can be approximated by

$$\frac{dZ_f}{dl} \sim -\frac{c_4 v_{\Delta}}{N v_F} Z_f \sim -\frac{c_5}{l \ln l} Z_f, \quad (31)$$

where $c_4 \approx 0.426$ and $c_5 = \frac{\pi^2 c_4}{8} \approx 0.523$. Solving this equation leads to the asymptotic behavior

$$Z_f \sim \frac{1}{(\ln l)^{c_5}} \rightarrow 0 \quad (32)$$

in the limit $l \rightarrow +\infty$. To proceed, it proves convenient to utilize the relationship between ω and l :

$$\omega = \omega_0 e^{-l}, \quad (33)$$

where ω_0 is a cutoff. Then the real part of retarded self-energy is approximated by

$$\text{Re } \Sigma_{\text{nem}}^R(\omega) \sim \omega \left\{ \ln \left[\ln \left(\frac{\omega_0}{\omega} \right) \right] \right\}^{c_5} \quad (34)$$

as $\omega \rightarrow 0$. Using the K.-K. relation, we obtain the imaginary part of self-energy

$$\text{Im } \Sigma_{\text{nem}}^R(\omega) \sim \frac{\pi}{2} \frac{|\omega|}{\ln \left(\frac{\omega_0}{\omega} \right) \left\{ \ln \left[\ln \left(\frac{\omega_0}{\omega} \right) \right] \right\}^{1-c_5}}. \quad (35)$$

From equation (35), it is easy to verify that this fermion damping rate is smaller than that in a MFL and manifests the asymptotic behavior $\lim_{\omega \rightarrow 0} \text{Im } \Sigma_{\text{nem}}^R(\omega)/\omega \rightarrow 0$, confirming the above analysis based on numerical results. According to the conventional notion of quantum many-body physics, one would expect the system to behave like a normal Fermi liquid. However, in the low energy limit $\omega \rightarrow 0$, the residue Z_f actually vanishes:

$$Z_f \sim \frac{1}{\left\{ \ln \left[\ln \left(\frac{\omega_0}{\omega} \right) \right] \right\}^{c_5}} \rightarrow 0, \quad (36)$$

which clearly implies that the system under consideration is actually a non-Fermi liquid and the nodal fermions do not have a well-defined quasiparticle peak in the spectral function. We see that the above residue Z_f flows to zero at a lower speed than that of a MFL, i.e., $Z_f \sim \frac{1}{\ln(\omega_0/\omega)}$. Therefore, the quantum critical fluctuation of nematic order gives rise to an even weaker violation of ordinary Fermi liquid theory than a MFL. To the best of our knowledge, this sort of unconventional non-Fermi liquid state has not been reported previously.

It is now interesting to compare this unconventional non-Fermi liquid state with graphene. In graphene, early perturbative calculations revealed that Dirac fermions exhibit MFL behavior with damping rate $\text{Im } \Sigma^R(\omega) \sim \omega$ and vanishing Z_f [52]. Nevertheless, subsequent careful RG studies [53–56] have found that, the fermion damping rate actually depends on energy as $\text{Im } \Sigma^R(\omega) \propto \omega/\ln^2 \omega$ at zero temperature due to the long-range Coulomb interaction, whereas the corresponding Z_f flows to a finite value as $\omega \rightarrow 0$. Therefore, graphene is a normal Fermi liquid. In contrast, the massless nodal fermions constitute a non-Fermi liquid at the nematic QCP, because Z_f vanishes in the lowest energy limit. The crucial difference between these two cases can be understood as follows. At nematic QCP, the fermion velocities are driven to vanish by the nematic order, so the effective fermion-nematic interaction is significantly pronounced. In graphene, however, the fermion velocity is dramatically enhanced at low energies by Coulomb interaction, which then weakens the effective Coulomb interaction and guarantees the validity of the Fermi liquid description. These two interacting Dirac fermion systems provide interesting new insight on the effects of strong electron correlations and also on the criterion of non-Fermi liquid states.

Moreover, an important lesson one can learn from the research experience of graphene is that RG may lead to qualitatively different spectral properties of fermions from that obtained by ordinary perturbative expansion approach. Indeed, this is the main motivation that has promoted us to make an extensive RG analysis of the spectral properties of nodal fermions at the nematic QCP.

3.2. Density of states and specific heat

We have showed in the last section that the nodal fermions exhibit unconventional non-Fermi liquid behaviors at the nematic QCP. In this section, we will compute two important quantities, namely DOS and specific heat, on the basis of the RG solutions with the goal to gain a better understanding of the non-Fermi liquid state.

The fermion DOS can be calculated from the retarded Green functions of nodal fermions via the definition [25]

$$\begin{aligned} \rho(\omega) &= \sum_{a=1}^N \int \frac{dk_x dk_y}{(2\pi)^2} \frac{1}{\pi} \text{Tr} [\text{Im } G_{1a}^R(v_F k_x, v_\Delta k_y, \omega) + \text{Im } G_{2a}^R(v_F k_x, v_\Delta k_y, \omega)] \\ &= \frac{1}{v_\Delta v_F} \sum_{a=1}^N \int \frac{dk'_x dk'_y}{(2\pi)^2} \frac{1}{\pi} \text{Tr} [\text{Im } G_{1a}^R(k'_x, k'_y, \omega) + \text{Im } G_{2a}^R(k'_x, k'_y, \omega)]. \end{aligned} \quad (37)$$

In the absence of interactions, the fermion DOS is well-known to be linear in ω , namely $\rho(\omega) \propto \omega$. This linear behavior will be changed once the interaction effects are considered. After including the fermion velocity renormalization, employing the method presented in [25], the flow equation for $\rho(\omega)$ is given by

$$\frac{d \ln \rho}{d \ln \omega} = \begin{cases} \frac{1 + C_1 - C_2 - C_3 - C_g}{1 - C_1 + C_2 + C_g}, & \text{if } v_\Delta < v_F, \\ \frac{1 + C_1 - C_2 - C_3 - C_g}{1 - C_1 + C_3 + C_g}, & \text{if } v_\Delta > v_F. \end{cases} \quad (38)$$

In cuprates, the initial value of velocity ratio $v_{\Delta 0}/v_{F0}$ is known to be much smaller than 1 [57]. Additionally, due to the quantum fluctuation of nematic order, v_Δ/v_F decreases monotonously as the energy scale is lowering.

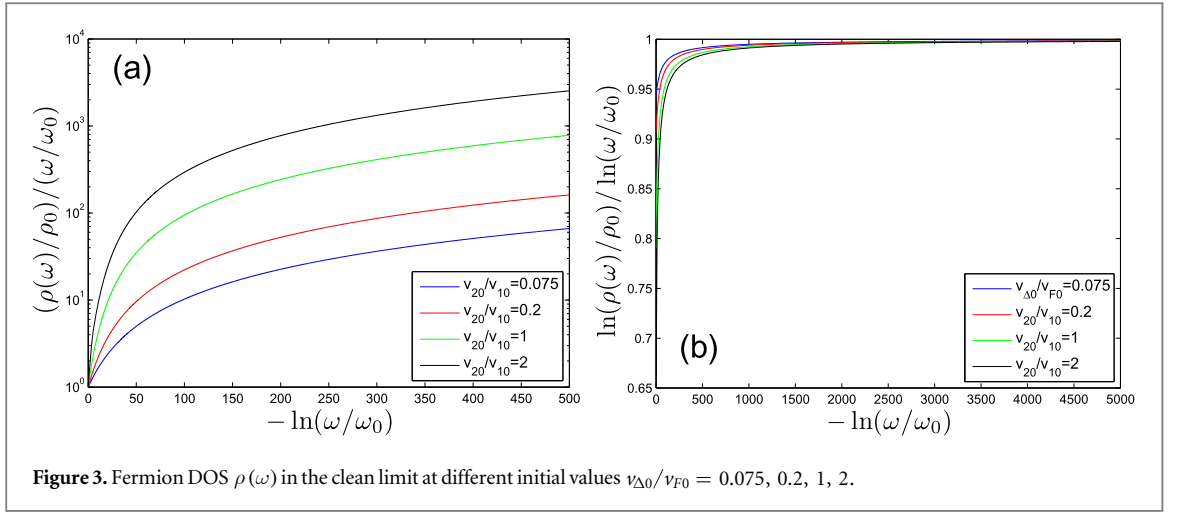


Figure 3. Fermion DOS $\rho(\omega)$ in the clean limit at different initial values $v_{\Delta 0}/v_{F0} = 0.075, 0.2, 1, 2$.

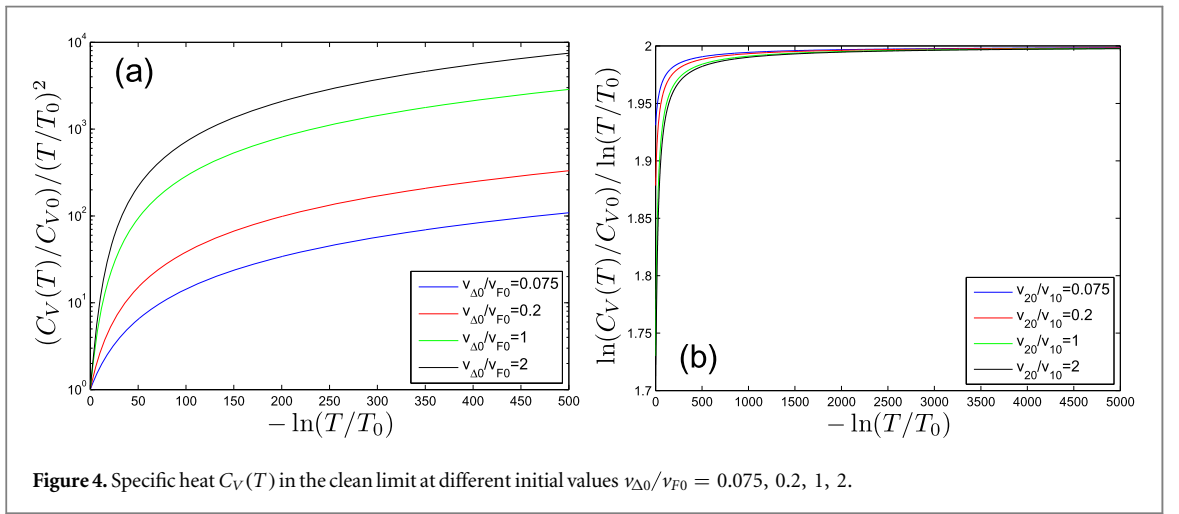


Figure 4. Specific heat $C_V(T)$ in the clean limit at different initial values $v_{\Delta 0}/v_{F0} = 0.075, 0.2, 1, 2$.

Therefore, for a given initial value $v_{\Delta 0}/v_{F0} < 1$, we only need to consider the case $v_{\Delta} < v_F$. In order to show that the conclusion is independent of the condition $v_{\Delta 0}/v_{F0} < 1$, we also plot the curves for the case $v_{\Delta 0}/v_{F0} = 2$ in figures 3 and 4. The RG equations of DOS and specific heat are also given with a generalized form. In the clean limit, we plot the results for $\rho(\omega)$ in figure 3. We see from figure 3(a) that $\rho(\omega)$ is apparently not linear in ω , but displays the asymptotic behavior

$$\lim_{\omega \rightarrow 0} \frac{\rho(\omega)}{\omega} \rightarrow +\infty. \quad (39)$$

On the other hand, figure 3(b) implies that

$$\lim_{\omega \rightarrow 0} \frac{\ln(\rho(\omega)/\rho_0)}{\ln(\omega/\omega_0)} \rightarrow 1. \quad (40)$$

This DOS $\rho(\omega)$ is qualitatively different from the power law function $\rho(\omega) \sim \omega^{1-\alpha}$, where α is a small finite value, obtained previously in [25].

In the non-interacting limit, the fermion specific heat depends on T as $C_V(T) \propto T^2$. As shown in B, including the influence of the quantum fluctuation of nematic order and random gauge potential leads us to

$$\frac{d \ln C_V}{d \ln T} = \begin{cases} 2 + \frac{2C_1 - C_2 - C_3 - 2C_g}{1 - C_1 + C_2 + C_g}, & \text{if } v_{\Delta} < v_F, \\ 2 + \frac{2C_1 - C_2 - C_3 - 2C_g}{1 - C_1 + C_3 + C_g}, & \text{if } v_{\Delta} > v_F. \end{cases} \quad (41)$$

At low T , $C_V(T)$ behaves as

$$\lim_{T \rightarrow 0} \frac{C_V(T)}{T^2} \rightarrow +\infty, \quad (42)$$

which is visualized in figure 4(a). Apparently, the original quadratic T -dependence of $C_V(T)$ obtained in the non-interacting limit is significantly altered. According to figure 4(b), we can express $C_V(T)$ at very small T as

$$\lim_{T \rightarrow 0} \frac{\ln(C_V(T)/C_{V0})}{\ln(T/T_0)} \rightarrow 2, \quad (43)$$

which is also distinct from the power law T -dependence $C_V(T) \sim T^{2-\beta}$, where β is a small finite constant, obtained previously [25].

Simple analysis reveal that the three parameters C_1 , C_2 , and C_3 appearing in the RG equations (38) and (41) all flow to zero in the lowest energy limit [24]. This asymptotic behavior makes it impossible to express $\rho(\omega)$ and $C_V(T)$ by power law functions. As given in appendix C, the low energy behavior of $\rho(\omega)$ and the low- T behavior of $C_V(T)$ can be approximately expressed as

$$\rho(\omega) \sim \left(\frac{\omega}{\omega_0}\right) \ln\left(\frac{\omega_0}{\omega}\right) \left(\ln\ln\left(\frac{\omega_0}{\omega}\right)\right)^{a_\rho} \exp\left[\frac{1}{2}\left(\ln\ln\ln\left(\frac{\omega_0}{\omega}\right)\right)^2\right], \quad (44)$$

$$C_V(T) \sim \left(\frac{T}{T_0}\right)^2 \ln\left(\frac{T_0}{T}\right) \left(\ln\ln\left(\frac{T_0}{T}\right)\right)^{a_C} \exp\left[\frac{1}{2}\left(\ln\ln\ln\left(\frac{T_0}{T}\right)\right)^2\right], \quad (45)$$

where a_ρ and a_C are two negative constants. At $N=2$, $a_\rho \approx -1.896$ and $a_C \approx -1.466$.

We notice that our RG equation for $C_V(T)$ (41) is not exactly the same as that obtained by Xu *et al* [25]. This difference does not affect our conclusion that $C_V(T)$ is not a power law function at low T . Indeed, if we start from the RG equation of $C_V(T)$ presented in [25], we reach the same conclusion. A more detailed discussion is presented in appendix B and appendix C.

4. Impact of random gauge potential

In this section, we investigate the impact of random gauge potential on the quantum critical behaviors near the nematic QCP. We will first show the RG solutions obtained in the presence of random gauge potential, and then re-calculate the fermion DOS and specific heat after considering the influence of disorder scattering. To this end, we need to retain a nonzero C_g in the RG equations (11)–(15), (27), (38), and (41), and then solve these RG equations numerically.

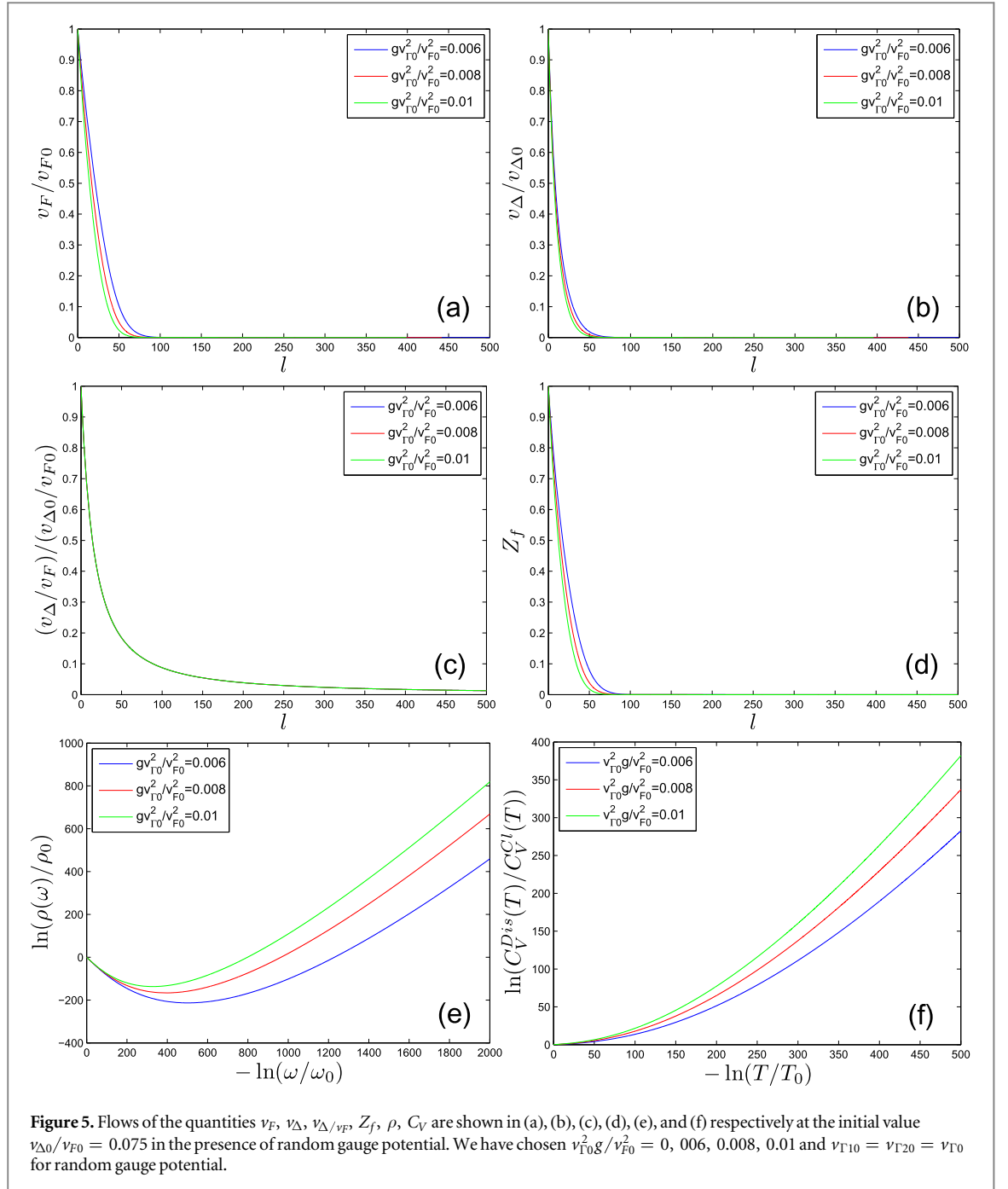
4.1. Flow of effective disorder strength

In order to make a direct comparison to the clean case in which $C_g = 0$ and to explicitly see the influence of a nonzero C_g on the running behaviors of various parameters, we plot the l -dependence of v_F , v_Δ , v_Δ/v_F , and Z_f in figures 5(a)–(d). Moreover, we show the ω -dependence of $\rho(\omega)$ in figure 5(e) and the T -dependence of C_V in figure 5(f), respectively. We will discuss under what circumstances these RG results are modified in the next section.

Comparing figures 3(a) and (b) with figures 5(a) and (b), we see that the detailed l -dependence of v_F and v_Δ are both altered dramatically by the random gauge potential. As shown in figure 3(c) and figure 5(c), the velocity ratio v_Δ/v_F exhibits exactly the same l -dependence in the clean and disordered cases, which originates from the fact that C_g does not enter into the RG equation of velocity ratio v_Δ/v_F . According to figure 2(a) and figure 5(d), the renormalization factor Z_f flows to zero in the disordered case much more rapidly than the clean case.

As shown in figure 5(e), the DOS $\rho(\omega)$ is divergent in the lowest energy limit due to random gauge potential. This is completely different from the behaviors of clean case presented in figure 3. In figure 5(f), we plot the ratio between $C_V^{\text{Dis}}/C_V^{\text{Cl}}$, where C_V^{Dis} and C_V^{Cl} are the specific heat obtained in disordered and clean cases, respectively. Since the solutions of the RG equations about DOS and specific heat are modified substantially in the disordered case, it turns out that random gauge potential is a relevant perturbation in the present system. To verify the relevance of random gauge potential, we should appeal to the RG analysis of the effective disorder strength.

In the action term S_{dis} given in equation (4), the parameters that characterize the fermion-disorder coupling seem to be v_{Γ_1} and v_{Γ_2} . It can be seen from the RG equations that v_{Γ_1} flows in precisely the same way as v_F , and that v_{Γ_2} as v_Δ . Thus, both v_{Γ_1} and v_{Γ_2} are strongly renormalized and driven to vanish as $l \rightarrow +\infty$. However, this does not mean that the disorders can be simply neglected. Indeed, whether disorders are important is determined by the ratio between the interaction energy given by S_{dis} and the fermion energy $E_k \propto (v_F^2 k_x^2 + v_\Delta^2 k_y^2)^{1/2}$. We can see that this ratio is defined by C_g , which enters into the RG equations for the parameters v_F , v_Δ , v_{Γ_1} , and v_{Γ_2} . The effective strength of random gauge potential should be measured by C_g , rather than v_{Γ_1} and v_{Γ_2} . Recall that C_g is a function of five parameters, i.e., v_F , v_Δ , v_{Γ_1} , v_{Γ_2} , and g . Among these parameters, g is assumed to be a dimensionless constant, but the other four parameters flow strongly with the varying l . Detailed RG analysis revealed that C_g goes to infinity, namely



$$C_g = \frac{g}{2\pi} \left(\frac{v_{\Gamma 10}^2 v_F}{v_{F0}^2 v_{\Delta}} + \frac{v_{\Gamma 20}^2 v_{\Delta}}{v_{\Delta 0}^2 v_F} \right) \rightarrow +\infty, \quad (46)$$

in the limit $l \rightarrow +\infty$ as a consequence of the singular renormalization of the fermion velocities ratio v_{Δ}/v_F . More quantitatively, the large scale behavior of C_g can be described by

$$C_g \sim c_3 l \ln l, \quad (47)$$

where $c_3 = \frac{4g}{\pi^3 N} \frac{v_{\Gamma 10}^2}{v_{F0}^2}$.

It is necessary to explain here why the effective strength of random gauge potential should be represented by C_g , rather than solely by the coefficients $v_{\Gamma i}$ with $i = 1, 2$, and why $C_g \rightarrow \infty$ in the lowest energy limit. In an interacting fermion system, the effective strength of the interaction is characterized by the ratio between the interaction energy scale and the kinetic energy of fermions. This ratio is widely used in condensed matter physics to judge whether an interacting fermion system can be defined as a strongly correlated system or not. For instance, the normal metal with a high density of itinerant electrons is believed to be a weakly interacting system since the energy scale of Coulomb potential is much smaller than the Fermi energy. In a massless Dirac fermion

system (such as graphene), the effective strength of long-range Coulomb interaction is defined as $\alpha \sim \frac{e^2}{v_F}$, where e^2 appears in the action of the Coulomb interaction as a coupling coefficient and the fermion velocity v_F reflects the order of the kinetic energy [54, 56]. Another example comes from the effective BCS model of Dirac fermion systems, where the effective strength of pairing interaction is characterized by $g \sim \frac{u}{v_F}$ with u being the coupling coefficient of pairing interaction and v_F fermion velocity [58, 59].

This criterion also applies to disordered systems. When massless Dirac fermions couple to random gauge potential, the effective strength of random gauge potential should be defined by $C_g \sim \frac{v_{\Gamma i}^2}{v_F v_{\Delta}}$, rather than the coefficient of fermion-disorder coupling $v_{\Gamma i}$ [47, 48]. When $v_{\Gamma i}$ flows to zero in the lowest energy limit, C_g does not necessarily vanish since there is a possibility that v_F and v_{Δ} may vanish more (or equally) rapidly than $v_{\Gamma i}$. When the nodal fermions couple to both the quantum fluctuation of nematic order and random gauge potential, the four parameters v_F , v_{Δ} , $v_{\Gamma 1}$, and $v_{\Gamma 2}$ all flow to zero in the lowest energy limit, but the effective strength of random gauge potential becomes very large, namely $C_g \rightarrow \infty$. This originates from the fact that $\frac{v_{\Gamma 1}}{v_F} = \frac{v_{\Gamma 10}}{v_{F0}}$ and $\frac{v_{\Gamma 2}}{v_{\Delta}} = \frac{v_{\Gamma 20}}{v_{\Delta 0}}$ are constants, while at the same time the velocity ratio $v_{\Delta}/v_F \rightarrow 0$. The behavior $C_g \rightarrow \infty$ at low energies indicates that random gauge potential is relevant. To see this point, we neglect the nematic order and consider only the coupling of nodal fermions to random gauge potential, which results in simplified RG equations:

$$\frac{dv_F}{dl} = -C_g v_F, \quad \frac{dv_{\Delta}}{dl} = -C_g v_{\Delta}, \quad \frac{d(v_{\Delta}/v_F)}{dl} = 0, \quad \frac{dv_{\Gamma 1}}{dl} = -C_g v_{\Gamma 1}, \quad \frac{dv_{\Gamma 2}}{dl} = -C_g v_{\Gamma 2}. \quad (48)$$

Using the two relations $\frac{v_{\Gamma 1}}{v_F} = \frac{v_{\Gamma 10}}{v_{F0}}$ and $\frac{v_{\Gamma 2}}{v_{\Delta}} = \frac{v_{\Gamma 20}}{v_{\Delta 0}}$, it is easy to show that

$$C_g = \frac{g}{2\pi} \frac{v_{\Gamma 1}^2 + v_{\Gamma 2}^2}{v_F v_{\Delta}} = \frac{g}{2\pi} \frac{v_{\Gamma 10}^2 + v_{\Gamma 20}^2}{v_{F0} v_{\Delta 0}} = C_{g0}. \quad (49)$$

In this case, v_F , v_{Δ} , $v_{\Gamma 1}$, and $v_{\Gamma 2}$ depend on length scale l as

$$\frac{v_F}{v_{F0}} = \frac{v_{\Delta}}{v_{\Delta 0}} = \frac{v_{\Gamma 1}}{v_{\Gamma 10}} = \frac{v_{\Gamma 2}}{v_{\Gamma 20}} = e^{-C_{g0}l}, \quad (50)$$

which all vanish rapidly in the limit $l \rightarrow \infty$. Since C_g is a constant, random gauge potential is marginal. From the above calculations, we can conclude that the behavior $C_g \rightarrow \infty$ obtained in the presence of both nematic fluctuation and random gauge potential is directly related to the extreme velocity anisotropy $v_{\Delta}/v_F \rightarrow 0$ induced by the quantum fluctuation of nematic order. The flow of C_g towards strong coupling regime is a clear signature that random gauge potential should have substantial physical effects on the low-energy behaviors of nodal fermions, which will be discussed in the next section.

We notice other interesting correlated electron models in which the coupling coefficient of an interaction vanishes at low energies, but the interaction is not negligible due to the more (or equally) rapid decrease of the kinetic energy of electrons. Recently, Sur and Lee studied the influence of quantum fluctuations of an antiferromagnetic (AF) order at an AF quantum critical point in a metal supporting one-dimensional Fermi surface [60]. In particular, they showed that the coupling coefficient of the interaction flows to zero at low energies. However, the fermion velocity also vanishes, thus the interaction cannot be simply neglected. Actually, Sur and Lee found that the interaction drives the system to become a so-called quasi-local strange metal that is apparently qualitatively different from the free fermion system.

4.2. Physical effects of random gauge potential

How should we understand the divergence of C_g ? In order to answer this question, we first consider the non-interacting system that contains only nodal fermions and random gauge potential. In this case, the RG equation of fermion DOS becomes

$$\frac{d \ln \rho}{d \ln \omega} = \frac{1 - C_{g0}}{1 + C_{g0}}, \quad (51)$$

with

$$C_{g0} = \frac{g}{2\pi} \frac{v_{\Gamma 10}^2 + v_{\Gamma 20}^2}{v_{F0} v_{\Delta 0}}. \quad (52)$$

If $C_{g0} < 1$, we have

$$\rho(\omega) \sim \omega^{\alpha} \quad \text{with} \quad \alpha = \frac{1 - C_{g0}}{1 + C_{g0}}, \quad (53)$$

where α satisfies $0 < \alpha < 1$. In this case, the RG equation for specific heat is given by

$$\frac{d \ln C_V}{d \ln T} = 2 - \frac{2C_{g0}}{1 + C_{g0}}. \quad (54)$$

The solution of this equation is given by

$$C_V(T) \sim T^\beta \quad \text{with} \quad \beta = 2 - \frac{2C_{g0}}{1 + C_{g0}}. \quad (55)$$

If $C_g > 1$, it is easy to get

$$\rho(\omega) \rightarrow \infty \quad (56)$$

in the limit $\omega \rightarrow 0$. The divergence $\rho(\omega)$ indicates the emergence of a disorder dominated diffusive state, in which a finite disorder scattering rate γ and a finite zero-energy DOS $\rho(0)$ are generated [49, 61, 62]. The value $C_g = 1$ defines a QCP for the quantum phase transition between the ballistic and diffusive states of nodal fermions. Therefore, a weak random gauge potential gives rise to power law behavior of $\rho(\omega)$, whereas a sufficiently strong disorder can trigger the quantum phase transition to a diffusive state.

In the presence of both nematic critical fluctuation and random gauge potential, the fact that $C_g \rightarrow \infty$ in the lowest energy limit signals the development of a disorder dominated diffusive state and the generation of a finite γ and a finite $\rho(0)$ even in the case of weak random gauge potential. Although the perturbative RG method provides a powerful tool to judge whether and when a phase transition takes place [63–65], it is not a suitable method to compute the precise value of γ . To calculate γ , one usually needs to construct a set of self-consistent equations for the retarded fermion self-energy by properly considering both the fermion-disorder scattering and the influence of quantum critical fluctuation of nematic order [66]. This is an interesting yet complicated issue, which is beyond the scope of the present work and subjected to future research. However, it is technically possible to make a rough estimation for the energy scale of γ by means of the RG method. As shown in figure 5(e), for small given values of C_{g0} and $v_{\Delta 0}/v_{F0}$, the solution of the RG equation of DOS should have the following properties: $\rho(\omega)$ decreases as the varying energy scale decreases, but tends to increase when the energy scale exceeds a critical value $E_c(C_{g0}, v_{\Delta 0}/v_{F0})$, which is a function of C_{g0} and $v_{\Delta 0}/v_{F0}$. The magnitude of γ should be an increasing function of E_c . In the following calculations of $\rho(0)$ and $C_V(T)$, we will regard γ as an undetermined constant. Fortunately, the qualitative behaviors of $\rho(0)$ and $C_V(T)$ in the low energy regime is independent of the precise value of γ .

The imaginary part of retarded fermion self-energy can be generically written as

$$\text{Im} \Sigma^R(\omega) \approx \gamma + \text{Im} \Sigma_{\text{nem}}^R(\omega), \quad (57)$$

where $\text{Im} \Sigma_{\text{nem}}^R(\omega)$ is the contribution induced solely by the nematic order. The disorder-induced scattering rate γ represents a characteristic energy scale. At energies higher than γ , namely $\omega > \gamma$, $\text{Im} \Sigma_{\text{nem}}^R(\omega)$ dominates over γ and all the RG results for v_F , v_Δ , Z_f , $\rho(\omega)$, and $C_V(T)$ shown in figures 5(a)–(f) are still applicable. At $\omega < \gamma$, the diffusive motion of nodal fermions and its interplay with critical nematic fluctuation govern the low-energy properties of the system.

Once a finite γ is generated, the renormalized velocities v_F and v_Δ no more vanish at low energies, which is apparently different from the clean case. Instead, as the energy scale decreases, both v_F and v_Δ are saturated to certain finite values, denoted by v'_{F0} and $v'_{\Delta 0}$, below the energy scale set by γ . Hence, there is no extreme velocity anisotropy in the diffusive state. The fermion DOS and specific heat also exhibit different behaviors comparing to the clean case. To demonstrate the difference in DOS, we write the retarded propagators of nodal fermions in the forms:

$$G_{1a}^R(\omega, \mathbf{k}) = \frac{1}{-(\omega + i\gamma) + v'_{F0}k_x\tau^z + v'_{\Delta 0}k_y\tau^x}, \quad (58)$$

$$G_{2a}^R(\omega, \mathbf{k}) = \frac{1}{-(\omega + i\gamma) + v'_{F0}k_y\tau^z + v'_{\Delta 0}k_x\tau^x}. \quad (59)$$

Calculations find that the fermion DOS depends on ω as

$$\rho(\omega) = \frac{N\gamma}{2\pi v'_{F0}v'_{\Delta 0}} \left[\ln \left(\frac{(\gamma^2 - \omega^2 + v'_{F0}v'_{\Delta 0}\Lambda^2)^2 + 4\omega^2\gamma^2}{(\gamma^2 - \omega^2)^2 + 4\omega^2\gamma^2} \right) + \frac{2\omega}{\gamma} \arctan \left(\frac{\gamma^2 - \omega^2 + v'_{F0}v'_{\Delta 0}\Lambda^2}{2\omega\gamma} \right) - \frac{2\omega}{\gamma} \arctan \left(\frac{\gamma^2 - \omega^2}{2\omega\gamma} \right) \right]. \quad (60)$$

In the case $\sqrt{v'_{F0}v'_{\Delta 0}}\Lambda \gg \gamma$, we have

$$\rho(0) \approx \frac{2N\gamma}{\pi v'_{F0}v'_{\Delta 0}} \ln \left(\frac{\sqrt{v'_{F0}v'_{\Delta 0}}\Lambda}{\gamma} \right). \quad (61)$$

In order to compute the specific heat, it is convenient to invoke the standard Matsubara formalism of fermion propagators, i.e.,

$$G_{1a}(\omega_n, \mathbf{k}) = \frac{1}{-i(\omega_n + \gamma \text{sgn}(\omega_n)) + v'_{F0}k_x\tau^z + v'_{\Delta 0}k_y\tau^x},$$

$$G_{2a}(\omega_n, \mathbf{k}) = \frac{1}{-i(\omega_n + \gamma \text{sgn}(\omega_n)) + v'_{F0}k_y\tau^z + v'_{\Delta 0}k_x\tau^x},$$

where $\omega_n = (2n + 1)\pi T$ with n being integer. The fermion free energy is given by

$$F = -\frac{2NT}{v'_{F0}v'_{\Delta 0}} \sum_{\omega_n} \int \frac{d^2k}{(2\pi)^2} \ln [(\omega_n + \gamma \text{sgn}(\omega_n))^2 + k^2].$$

Summing over ω_n leads to

$$F \approx -\frac{4N}{\pi v'_{F0}v'_{\Delta 0}} \int \frac{d^2k}{(2\pi)^2} \left\{ k \arctan \left(\frac{k}{\pi T + \gamma} \right) + \frac{\gamma}{2} \ln [(\pi T + \gamma)^2 + k^2] \right\}. \quad (62)$$

Now the fermion specific heat in the low energy regime can be approximately given by

$$C_V \approx \frac{6NT}{v'_{F0}v'_{\Delta 0}} \gamma \ln \left(\frac{\Lambda}{\gamma} \right), \quad (63)$$

which depends on T linearly. We can see that DOS and specific heat obtained in the diffusive state exhibit entirely different behaviors from the unconventional non-Fermi liquid state below the energy scale γ .

We finally make a brief remark on the behavior of the system staying away from the nematic QCP. Suppose the system stays at a distance r to the QCP, the RG results are still valid and the fermion velocity ratio is still renormalized at energies higher than the scale corresponding to r . However, the renormalization terminates at certain low energy scale. Therefore, the effective strength of random gauge potential is moderately enhanced compared its bare value, though C_g does not diverge.

5. Comparison with experiments

In this section, we address the possible connection between the theoretical results obtained in the last sections and the phenomenology of cuprate superconductors. We are particularly interested in three existing experimental findings about some of the unusual properties of the superconducting dome.

5.1. Anomalous residual linear- T term of specific heat in cuprates

We first discuss the residual specific heat in cuprates. Due to the line nodes of d -wave gap, the specific heat in the superconducting phase of cuprates is expected to exhibit a T^2 behavior, i.e., $C_V(T) \propto T^2$, at low T . This expectation is generically consistent with experiments [67]. In the lowest T limit, experiments have observed a residual linear- T term of $C_V(T)$ [67–70], which is usually attributed to the finite fermion DOS generated by disorder scattering and is also well consistent with the result given by equation (63). There is, however, an unexpected experimental finding [67–70] that the residual specific heat of YBCO is obviously larger in magnitude than that of $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$ (LSCO), although the former material is known to be cleaner than the latter. Apparently, disorder scattering alone is not capable of accounting for this experimental fact. Recently, coexistence of d -wave superconductivity with a loop current order was proposed to give a possible explanation [71–73] for the large residual linear- T term of specific heat in YBCO. In this scenario, when d -wave superconductivity coexists with a loop current order, two of four nodal points are converted to finite Fermi pockets of Bogoliubov quasiparticles, which then generates a finite $\rho(0)$ and a residual linear- T term of specific heat [71–74]. The recent ultrasound measurements showed the possible evidence for the existence of loop current order in YBCO [75, 76].

Here we propose an alternative explanation for the above anomalous experimental results of residual specific heat. Our RG analysis found that the effective strength of random gauge potential, being the most relevant disorder to cuprates [47–49], is strongly enhanced by the critical nematic fluctuation, which in turn increases the residual value of the specific heat. To understand the role of quantum nematic fluctuation, we firstly consider only the coupling of nodal fermions to random gauge potential. In this special case, C_g does not flow and thus

remains a constant, namely $C_g = C_{g0}$. If C_{g0} is very small, which means the system is only slightly disordered, the behavior of $\rho(\omega)$ and specific heat would be governed by equations (53) and (55), respectively. In this case, the system does not develop a finite $\rho(0)$ and there is no residual linear- T term of specific heat. If the system is quite disordered such that $C_{g0} > 1$, it enters a diffusive state with a finite scattering rate γ , which induces a finite $\rho(0)$ and a residual linear- T term of specific heat. Apparently, the residual specific heat is larger in more disordered systems, which is not consistent with the aforementioned experiments of residual specific heat. If we consider both the quantum fluctuation of nematic order and random gauge potential, C_g is significantly enhanced and flows to strong coupling at low energies even if C_{g0} takes an arbitrarily small value. This implies that a cleaner compound might acquire a larger amount of γ and naturally larger $\rho(0)$ and residual linear- T term of specific heat.

To make a more careful comparison between theories and experiments, we now briefly discuss the doping dependence of our results. We use δ to denote the doping concentration and δ_c the nematic QCP. At zero temperature, the mass of nematic field ϕ is proportional to the difference between δ and δ_c , namely $r \sim |\delta - \delta_c|$. When the cuprate is at a distance r away from the nematic QCP, the quantum fluctuation of nematic order is not critical, but remains important for small r . At the energy scales larger than r , the fermion velocity anisotropy is still considerably enhanced, which leads C_g to flow to larger values. For a given small value of C_{g0} , C_g can flow to a sufficiently large value to induce a diffusive state and generate a finite scattering rate γ , provided that r is made sufficiently small. In this case, the quantum fluctuation of nematic order can result in a finite $\rho(0)$ and a finite residual linear- T term of specific heat. If the bare value C_{g0} is large enough, random gauge potential itself suffices to generate a finite γ . Including the quantum fluctuation of nematic order leads to larger values of both C_g and γ . In any case, the quantum fluctuation of nematic order tends to amplify γ , which naturally increase $\rho(0)$ and the residual specific heat.

A number of recent experiments provided strong evidence supporting the existence of nematic order in YBCO, whereas there is little evidence for nematicity in LSCO. Although YBCO is cleaner than LSCO, the quantum fluctuation of nematic order in the former superconductor can drive C_g to flow to much larger values. As a consequence, the residual linear-in- T term of specific heat of YBCO would be larger than that of LSCO. We emphasize that, disorder itself cannot explain the anomalous behaviors of the residual specific heat observed in YBCO and LSCO, and it is necessary to consider the interplay of quantum nematic fluctuation and random gauge potential.

From the above elaboration, we know that the roles played by the quantum nematic fluctuation and random gauge potential depends on doping δ . To simplify discussion, we assume the bare value C_{g0} displays only a weak δ -dependence. Since the quantum fluctuation of nematic order is most pronounced at the nematic QCP, the scattering rate γ and consequently the coefficient of the residual linear- T term of specific heat are maximal at the QCP, and decrease as the system moves away from this QCP. This doping dependence is observable, and can be examined by experiments. Within the loop current order scenario, both the zero-energy DOS $\rho(0)$ and the residual linear- T term of specific heat are proportional to the order parameter of loop current order [71–73], which may increase with lowering doping in the underdoped region.

Early experiments found that the coefficient of linear- T term of specific heat in optimally doped YBCO is roughly $2\text{mJ}\cdot\text{mol}^{-1}\cdot\text{K}^{-2}$ [67–69]. Recent measurements performed in underdoped YBCO revealed that this coefficient is $1.85 \pm 0.06\text{mJ}\cdot\text{mol}^{-1}\cdot\text{K}^{-2}$ [70]. We feel that the currently available experimental data about the doping dependence of the coefficient are still quite limited. We expect more extensive measurements would be performed in the future to extract a more quantitative doping dependence of the coefficient, which could help to judge whether the scenario proposed in this paper works.

5.2. Strong damping rate of nodal fermions in optimally doped BSCCO

We next apply the RG results to understand the observed damping rate of nodal fermions in cuprates. Valla *et al* [77] have performed extensive angle resolved photoemission spectroscopy measurements in optimally doped BSCCO. Their main discovery is that the nodal fermions exhibit a MFL-type damping rate in the normal state above T_c , which is in general consistency with the observed linear resistivity. They further found [77] that the linear damping rate is not sensitive to the onset of superconductivity and persists well below T_c . This was out of expectation since previous BCS weak coupling analysis [1, 78] had predicted a quite weak damping rate, i.e., $\text{Im}\Sigma^R(\omega, T) \propto \max(\omega^3, T^3)$, in the superconducting phase. Several scenarios [1, 20, 79] were proposed to account for the nearly MFL behavior. In particular, Vojta *et al* [20] and Khveshchenko and Paaske [79] have argued that the strong fermion damping may arise from a secondary phase transition from a $d_{x^2-y^2}$ -wave superconducting state to a $d_{x^2-y^2} + i$ or $d_{x^2-y^2} + id_{xy}$ -wave superconducting state.

Experimentally, the existence of a nematic phase was observed in BSCCO by Lawler *et al* [13]. More recent experimental studies of Fujita *et al* provided further evidence pointing towards the existence of a nematic QCP in the vicinity of optimal doping in BSCCO [14, 19]. Therefore, it seems natural to account for the experimental finding of Valla *et al* by considering the quantum critical fluctuation of nematic order. We have showed through

RG analysis that the nodal fermion damping rate caused by critical nematic fluctuation, described by $\text{Im} \Sigma_{\text{nem}}^R(\omega)$, is slightly weaker than that of a MFL. In realistic experiments, it is difficult to distinguish this non-Fermi liquid state from a MFL state. Therefore, presence of a nematic QCP provides an alternative scenario for the nearly MFL behavior observed in optimally doped BSCCO. However, the nearly MFL behavior occurs only at energies higher than the value ω_γ set by disorder scattering rate γ . Indeed, at $\omega < \omega_\gamma$, γ dominates over $\text{Im} \Sigma_{\text{nem}}^R(\omega)$, thus a finite zero-energy DOS $\rho(0)$ is generated. To summarize, our RG results are qualitatively consistent with the quasiparticle self-energy $\text{Im} \Sigma(\omega, T) = \gamma + \max(\omega, T)$ observed in [77].

5.3. Temperature dependence of fermion velocity in BSCCO

Another experiment is the observation of Plumb *et al* [80] that the fermion velocity v_F along the nodal directions increases as T grows in BSCCO. We know from RG results that the critical nematic fluctuations drive the fermion velocities to vanish in the lowest energy limit. This means the velocities must increase as the energy scale is growing. Therefore, the nematic quantum fluctuation will induce increment of fermion velocities when the thermal energy increases with growing T , which is qualitatively well consistent with the observation of Plumb *et al*. Technically, one can translate the l -dependence of fermion velocity to a T -dependence by making the transformation $T = T_0 e^{-l}$ [31, 81]. With the help of this transformation, it is easy to show that the fermion velocity v_F is an increasing function of T . Therefore, the singular renormalization of fermion velocities of nodal fermions induced by the quantum fluctuation of nematic order, which is first discovered by Huh and Sachdev [24], agrees with the T -dependence of v_F observed in [80].

6. Summary and discussions

In summary, we have found that the nodal fermions of d -wave superconductors constitute an unconventional non-Fermi liquid, which exhibits a weaker violation of Fermi liquid description than a MFL, due to the quantum critical fluctuation of nematic order. This unusual state represents a novel quantum state of matter that cannot be well captured by the traditional classification of (non-) Fermi liquids and thus enriches our knowledge of strong electron correlation effects. We also have calculated the fermion DOS and specific heat after incorporating the unusual renormalization of fermion velocities. When a gauge-potential-type disorder is added to the system, we analyzed its interplay with the quantum fluctuation of nematic order, and found that the effective disorder strength flows to strong coupling, leading to diffusive motion of nodal fermions. Therefore, even a weak random gauge potential can drive a quantum phase transition from an unconventional non-Fermi liquid state to a disorder dominated diffusive state. However, the unusual fermion damping induced by the nematic order is more important than disorder scattering at high temperatures, where the nodal fermions still display the unconventional non-Fermi liquid behaviors. We finally discussed the connection between our theoretic results and a number of interesting experiments in the context of some cuprate superconductors.

We now would like compare our work to the existing extensive works on the non-Fermi liquid behaviors in two-dimensional metals produced by the quantum critical fluctuation of nematic order. At the QCP of Pomeranchuk instability in two-dimensional metals, the quantum fluctuation of nematic order can lead to very strong fermion damping [36, 82–86]. To the leading order, it is found [36, 82–86] that the fermion damping rate behaves as $\text{Im} \Sigma(\omega) \sim \omega^{2/3}$ and the quasiparticle residue $Z_f \sim \omega^{1/3}$. Since Z_f vanishes in the limit $\omega \rightarrow 0$, this QCP exhibits non-Fermi liquid behavior. In this paper, we have considered the interaction between the quantum fluctuation of nematic order and massless nodal fermions in the superconducting dome of cuprate superconductors. It is apparent that the quantum critical fluctuation of nematic order gives rise to a stronger fermion damping effect in metals than in the superconducting dome of cuprates. This difference should be due to the different forms of the kinetic energies of fermionic excitations. In the context of cuprates, the kinetic energy of the massless Dirac fermions excited from the superconducting gap nodes is $E = \sqrt{v_F^2 k_x^2 + v_\Delta^2 k_y^2}$. In contrast, in a two-dimensional metal with a finite Fermi surface, the kinetic energy of fermions can be written as $E = v_F k_x + \frac{k_y^2}{2m}$, where k_x is the momentum component perpendicular to the Fermi surface and k_y is the momentum component along the tangential direction [36, 82–86]. In the low-energy regime, the latter kinetic energy is smaller than the former for the same given values of momenta, which indicates that the interaction plays a more important role in the latter system than the former. To further demonstrate this difference, we consider the different roles of the long-range Coulomb interaction in a two-dimensional Dirac semimetal and a two-dimensional semi-Dirac semimetal. In a Dirac semimetal, the kinetic energy of Dirac fermions is simply $E = v_F k$ with $k = \sqrt{k_x^2 + k_y^2}$. RG analysis showed that the residue Z_f approaches a finite value at low energies, so the system is a normal Fermi liquid [53–56]. In a semi-Dirac semimetal, the kinetic energy of fermions is written as $E = \sqrt{v_F^2 k_x^2 + \frac{k_y^2}{4m^2}}$ [87, 88]. In this case, the long-range Coulomb interaction drives Z_f to vanish in the lowest energy limit, i.e., $Z_f \rightarrow 0$, which apparently implies the breakdown of Fermi liquid behavior [87]. Once

again, we see that the ratio between the interaction energy scale and the kinetic energy is a crucial quantity to determine the low-energy behaviors of an interacting fermion system.

The coupling of nodal fermions to the quantum fluctuation of AF order is also an interesting problem [7, 89]. Uemura [7] considered the coupling of nodal fermions to the (π, π) AF fluctuation, and suggested a possibility that the (π, π) AF fluctuation can connect two nodal charges in different hole pockets and then generate a bound state of two nodal charges. More recently, Pelissetto *et al* studied a number of different couplings between nodal fermions and AF fluctuations using RG method [89]. An interesting result is that, though most of these couplings are irrelevant, a nearly marginal coupling between nodal fermions and an effective AF-order induced nematic fluctuation emerges. This nearly marginal coupling is found [89] to result in a fermion damping rate that is nearly linear in energy or temperature.

The electronic nematic state has been observed not only in some cuprate superconductors, but also in a number of iron-based superconductors: 122 family, such as hole doped $\text{Ba}_{1-x}\text{K}_x\text{Fe}_2\text{As}_2$, electron doped $\text{Ba}(\text{Fe}_{1-x}\text{Co}_x)_2\text{As}_2$, and isovalent-doped $\text{BaFe}_2(\text{As}_{1-x}\text{P}_x)_2$; 111 family, such as NaFeAs ; 1111 family, such as LaFeAsO ; 11 family, such as FeSe [90, 91]. In most of these compounds, the nematic order emerges in accompany with a spin density wave (SDW) order. However, there are also exceptions. For instance, the nematic order is observed in FeSe without any evidence for SDW order [90–98]. Whether the nematic order observed in iron-based superconductors is generated by the fluctuation of SDW order or the orbital degrees of freedom is still in fierce debate [90–99]. Recent experimental studies have unambiguously showed that there is a QCP in the superconducting dome at the optimal doping of $\text{BaFe}_2(\text{As}_{1-x}\text{P}_x)_2$. This QCP may correspond to the critical point for a SDW order or nematic QCP [100, 101], and is expected to exhibit rich quantum critical phenomena. Moreover, there are clear evidences that the superconducting gap of $\text{BaFe}_2(\text{As}_{1-x}\text{P}_x)_2$ has nodal line points [100]. Since the quantum fluctuation of nematic order is peaked at zero momentum [31, 91, 102], the nodal fermions excited from the nodal line points might couple strongly to the quantum fluctuation of nematic order at the nematic QCP [31]. This coupling is physically analogous to the model considered in this work, and it would be interesting to study this coupling by means of RG method. The RG analysis performed in this work could be generalized to study the possible non-Fermi liquid behavior and disorder effects in the context of $\text{BaFe}_2(\text{As}_{1-x}\text{P}_x)_2$ and other iron based superconductors, where the multi-band effects and different gap symmetry need to be seriously taken into account.

Acknowledgments

GZL and JRW would like to thank Dr Jing Wang for very helpful discussions. We acknowledge the support by the National Natural Science Foundation of China under Grants No.11274286, No.11574285, No.11504379, and No.U1532267.

Appendix A. Expressions of $C_{1,2,3}$

The expressions of C_1 , C_2 , and C_3 that appear in equations (11)–(15) are given by

$$C_1 = \frac{2(v_\Delta/v_F)}{N\pi^3} \int_{-\infty}^{\infty} dx \int_0^{2\pi} d\theta \frac{x^2 - \cos^2 \theta - (v_\Delta/v_F)^2 \sin^2 \theta}{(x^2 + \cos^2 \theta + (v_\Delta/v_F)^2 \sin^2 \theta)^2} \mathcal{G}(x, \theta), \quad (\text{A.1})$$

$$C_2 = \frac{2(v_\Delta/v_F)}{N\pi^3} \int_{-\infty}^{\infty} dx \int_0^{2\pi} d\theta \frac{-x^2 + \cos^2 \theta - (v_\Delta/v_F)^2 \sin^2 \theta}{(x^2 + \cos^2 \theta + (v_\Delta/v_F)^2 \sin^2 \theta)^2} \mathcal{G}(x, \theta), \quad (\text{A.2})$$

$$C_3 = \frac{2(v_\Delta/v_F)}{N\pi^3} \int_{-\infty}^{\infty} dx \int_0^{2\pi} d\theta \frac{x^2 + \cos^2 \theta - (v_\Delta/v_F)^2 \sin^2 \theta}{(x^2 + \cos^2 \theta + (v_\Delta/v_F)^2 \sin^2 \theta)^2} \mathcal{G}(x, \theta), \quad (\text{A.3})$$

with

$$\mathcal{G}^{-1} = \frac{x^2 + \cos^2 \theta}{\sqrt{x^2 + \cos^2 \theta + (v_\Delta/v_F)^2 \sin^2 \theta}} + \frac{x^2 + \sin^2 \theta}{\sqrt{x^2 + \sin^2 \theta + (v_\Delta/v_F)^2 \cos^2 \theta}}. \quad (\text{A.4})$$

Appendix B. RG equation for specific heat

The free energy density $\mathcal{F} = T \ln \mathcal{Z}/V$ is formally given by

$$\mathcal{F} = (\xi_\tau \xi_x \xi_y)^{-1}, \quad (\text{B.1})$$

where $\xi_\tau \sim \frac{1}{T}$, $\xi_x = v_F \xi_\tau$, and $\xi_y = v_\Delta \xi_\tau$. We can rewrite \mathcal{F} as

$$\mathcal{F} \sim \frac{1}{v_F v_\Delta} T^3, \quad (\text{B.2})$$

which yields the following specific heat:

$$C_V = -T \frac{\partial^2 \mathcal{F}}{\partial T^2} \sim \frac{1}{v_F v_\Delta} T^2. \quad (\text{B.3})$$

It is then easy to get

$$\frac{d \ln C_V}{d \ln T} = 2 + \frac{d \ln \left(\frac{1}{v_F v_\Delta} \right)}{d \ln T}. \quad (\text{B.4})$$

At a given T , the corresponding momentum scale should be determined by the larger component of the fermion velocities [25], i.e.,

$$\tilde{p} = \frac{T}{\max(v_F, v_\Delta)}, \quad (\text{B.5})$$

which then gives rise to

$$\frac{d \ln T}{d \ln \tilde{p}} = 1 + \frac{d \ln \max(v_F, v_\Delta)}{d \ln \tilde{p}}. \quad (\text{B.6})$$

The scaling equation for C_V is converted to

$$\frac{d \ln C_V}{d \ln T} = 2 + \frac{d \ln \left(\frac{1}{v_F v_\Delta} \right)}{d \ln \tilde{p} \frac{d \ln T}{d \ln \tilde{p}}} = 2 + \frac{d \ln \left(\frac{1}{v_F v_\Delta} \right)}{d \ln \tilde{p} \left(1 + \frac{d \ln \max(v_F, v_\Delta)}{d \ln \tilde{p}} \right)}. \quad (\text{B.7})$$

Since $d \ln \tilde{p} \sim -dl$, we find that

$$\frac{d \ln C_V}{d \ln T} = 2 + \frac{\frac{d \ln v_F}{dl} + \frac{d \ln v_\Delta}{dl}}{\left(1 - \frac{d \ln \max(v_F, v_\Delta)}{dl} \right)}. \quad (\text{B.8})$$

Substituting equations (11) and (12) into (B.8) leads us to

$$\frac{d \ln C_V}{d \ln T} = \begin{cases} 2 + \frac{2C_1 - C_2 - C_3 - 2C_g}{1 - C_1 + C_2 + C_g}, & \text{if } v_\Delta/v_F < 1, \\ 2 + \frac{2C_1 - C_2 - C_3 - 2C_g}{1 - C_1 + C_3 + C_g}, & \text{if } v_\Delta/v_F > 1. \end{cases} \quad (\text{B.9})$$

The quantum fluctuation of nematic order drives the velocity ratio v_Δ/v_F to monotonously decrease as one goes to lower and lower energies. It is known that the bare ratio $v_{\Delta 0}/v_{F 0} < 1$ in cuprate superconductors [1], which make it possible to simplify the RG equation to

$$\frac{d \ln C_V}{d \ln T} = 2 + \frac{2C_1 - C_2 - C_3 - 2C_g}{1 - C_1 + C_2 + C_g}. \quad (\text{B.10})$$

In the clean limit, $C_g = 0$, so the equation becomes

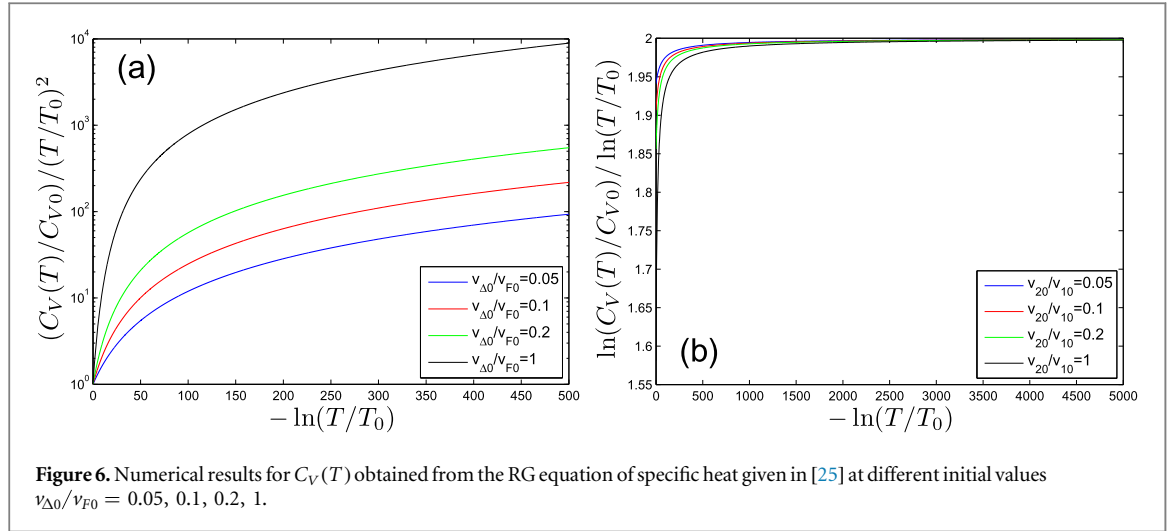
$$\frac{d \ln C_V}{d \ln T} = 2 + \frac{2C_1 - C_2 - C_3}{1 - C_1 + C_2}. \quad (\text{B.11})$$

This equation is slightly different from the RG equation for specific heat presented in [25], where the equation is

$$\frac{d \ln C_V}{d \ln T} = \frac{2 + 2C_1 - C_2 - C_3}{1 - C_1 + C_2}. \quad (\text{B.12})$$

The numerical solutions for equation (B.12) are shown in figure 6.

We notice that the specific heat obtained from equation (B.12) also satisfies $\lim_{T \rightarrow 0} C_V(T)/T^2 \rightarrow \infty$ and $\lim_{T \rightarrow 0} \ln(C_V(T))/\ln(T) \rightarrow 2$, which indicates that the specific heat is enhanced comparing to the case of non-interacting nodal fermion system. However, as will be shown in appendix C, it cannot be expressed as a power law function. The reason is that the three parameters C_1 , C_2 , and C_3 appearing in equation (B.12) all vanish in the lowest energy limit, rather than approaching certain finite values.



Appendix C. Approximate analytical expressions of DOS and specific heat

In order to show that the DOS and specific heat in clean limit do not exhibit power law behaviors, we now derive their approximate analytical expressions in the low-energy regime from the RG results. The RG equation of DOS is given by

$$\frac{d \ln \rho}{d \ln \omega} = \frac{1 + C_1 - C_2 - C_3}{1 - C_1 + C_2} \quad (\text{C.1})$$

in the absence of disorders. In the lowest energy limit, we know that the velocity ratio $v_{\Delta}/v_F \rightarrow 0$. The parameters C_1 , C_2 , and C_3 , being functions of v_{Δ}/v_F , all vanish [25] as $v_{\Delta}/v_F \rightarrow 0$. Thus the RG equation can be approximately written as

$$\begin{aligned} \frac{d \ln \rho}{d \ln \omega} &\approx 1 + 2C_1 - 2C_2 - C_3 \\ &\approx 1 - \frac{2}{N} \frac{v_{\Delta}}{v_F} a_1 + \frac{2}{N} \frac{v_{\Delta}}{v_F} a_2 - \left(\frac{8}{\pi^2} \ln \left(\frac{v_F}{v_{\Delta}} \right) - a_3 \right) \frac{1}{N} \frac{v_{\Delta}}{v_F} \\ &\approx 1 - (2a_1 - 2a_2 - a_3) \frac{1}{N} \frac{v_{\Delta}}{v_F} - \frac{8}{N\pi^2} \ln \left(\frac{v_F}{v_{\Delta}} \right) \frac{v_{\Delta}}{v_F}, \end{aligned} \quad (\text{C.2})$$

where we have used three constants $a_1 \approx 0.426$, $a_2 \approx 0.348$, and $a_3 \approx 0.96$ [24]. Substituting the approximate low-energy expression of v_{Δ}/v_F given in equation (22) into (C.2), we obtain

$$\begin{aligned} \frac{d \ln \rho}{d \ln \omega} &\sim 1 - (2a_1 - 2a_2 - a_3) \frac{1}{N} \frac{\pi^2 N}{8} \frac{1}{l \ln(l)} - \frac{8}{N\pi^2} \ln \left(\frac{8}{\pi^2 N} l \ln(l) \right) \frac{\pi^2 N}{8} \frac{1}{l \ln(l)} \\ &\sim 1 - a_p \frac{1}{l \ln(l)} - \frac{1}{l} - \frac{\ln \ln(l)}{l \ln(l)}, \end{aligned} \quad (\text{C.3})$$

where $a_p = (2a_1 - 2a_2 - a_3) \frac{\pi^2}{8} + \ln \left(\frac{8}{\pi^2 N} \right)$. For $N = 2$, $a_p \approx -1.896$. Using the relationship $\omega = \omega_0 e^{-l}$, we can solve equation (C.3) and then obtain the following analytical expression

$$\rho(\omega) \sim \left(\frac{\omega}{\omega_0} \right) \ln \left(\frac{\omega_0}{\omega} \right) \left(\ln \ln \left(\frac{\omega_0}{\omega} \right) \right)^{a_p} \exp \left[\frac{1}{2} \left(\ln \ln \ln \left(\frac{\omega_0}{\omega} \right) \right)^2 \right], \quad (\text{C.4})$$

which is applicable for small ω .

The RG equation for specific heat is shown in equation (B.11). At low energies, it can be approximated as

$$\begin{aligned} \frac{d \ln C_V}{d \ln T} &\sim 2 + 2C_1 - C_2 - C_3 \\ &\sim 2 - \frac{2}{N} \frac{v_{\Delta}}{v_F} a_1 + \frac{1}{N} \frac{v_{\Delta}}{v_F} a_2 - \left(\frac{8}{\pi^2} \ln \left(\frac{v_F}{v_{\Delta}} \right) - a_3 \right) \frac{1}{N} \frac{v_{\Delta}}{v_F} \\ &\sim 2 - a_C \frac{1}{l \ln(l)} - \frac{1}{l} - \frac{\ln \ln(l)}{l \ln(l)}, \end{aligned} \quad (\text{C.5})$$

where $a_C = (2a_1 - a_2 - a_3) \frac{\pi^2}{8} + \ln\left(\frac{8}{\pi^2 N}\right)$. At $N = 2$, $a_C \approx -1.466$. Using the transformation $T = T_0 e^{-l}$, we find that the specific heat at low temperature can be well approximated by the expression

$$C_V(T) \sim \left(\frac{T}{T_0}\right)^2 \ln\left(\frac{T_0}{T}\right) \left(\ln\ln\left(\frac{T_0}{T}\right)\right)^{a_C} \exp\left[\frac{1}{2} \left(\ln\ln\ln\left(\frac{T_0}{T}\right)\right)^2\right]. \quad (\text{C.6})$$

By applying the same treatment, the RG equation of specific heat presented in [25], shown in equation (B.12), can be approximately expressed as

$$C_V(T) \sim \left(\frac{T}{T_0}\right)^2 \ln\left(\frac{T_0}{T}\right) \left(\ln\ln\left(\frac{T_0}{T}\right)\right)^{a'_C} \exp\left[\frac{1}{2} \left(\ln\ln\ln\left(\frac{T_0}{T}\right)\right)^2\right], \quad (\text{C.7})$$

where $a'_C = (4a_1 - 3a_2 - a_3) \frac{\pi^2}{8} + \ln\left(\frac{8}{\pi^2 N}\right)$. At $N = 2$, $a'_C \approx -1.273$. The above two functions do not exhibit power law dependence on temperature.

References

- [1] Orenstein J and Millis A J 2000 *Science* **288** 468
- [2] Lee P A, Nagaosa N and Wen X-G 2006 *Rev. Mod. Phys.* **78** 17
- [3] Keimer B, Kivelson S A, Norman M R, Uchida S and Zaenen J 2015 *Nature* **518** 179
- [4] Tsuei C C and Kirtley J R 2000 *Rev. Mod. Phys.* **72** 969
- [5] Damascelli A, Hussain Z and Shen Z-X 2003 *Rev. Mod. Phys.* **75** 473
- [6] Norman M R 2005 *Adv. Phys.* **54** 715
- [7] Uemura Y J 2004 *J. Phys.: Condens. Matter* **16** S4515
- [8] Norman M R 2011 *Science* **332** 196
- [9] Scalapino 2012 *Rev. Mod. Phys.* **84** 1383
- [10] Ando Y, Segawa K, Komiya S and Lavrov A N 2002 *Phys. Rev. Lett.* **88** 137005
- [11] Hinkov V, Haug D, Fauquéx B, Bourges P, Sidis Y, Ivanov A, Bernhard C, Lin C T and Keimer B 2008 *Science* **319** 597
- [12] Daou R et al 2010 *Nature* **463** 519
- [13] Lawler M J et al 2010 *Nature* **466** 347
- [14] Fujita K et al 2014 *Science* **344** 612
- [15] Kivelson S A, Fradkin E and Emery V J 1998 *Nature* **393** 550
- [16] Kivelson S A, Bindloss I P, Fradkin E, Oganessian V, Tranquada J M, Kapitulnik A and Howald C 2003 *Rev. Mod. Phys.* **75** 1201
- [17] Vojta M 2009 *Adv. Phys.* **58** 699
- [18] Fradkin E, Kivelson S A, Lawler M J, Eisenstein J P and Mackenzie A P 2010 *Annu. Rev. Condens. Matter Phys.* **1** 153
- [19] Fradkin E, Kivelson S A and Tranquada J M 2015 *Rev. Mod. Phys.* **87** 457
- [20] Vojta M, Zhang Y and Sachdev S 2000 *Phys. Rev. Lett.* **85** 4940
- [21] Vojta M, Zhang Y and Sachdev S 2000 *Phys. Rev. B* **62** 6721
- [22] Vojta M, Zhang Y and Sachdev S 2000 *Int. J. Mod. Phys. B* **14** 3719
- [23] Kim E-A, Lawler M J, Oretto P, Sachdev S, Fradkin E and Kivelson S A 2008 *Phys. Rev. B* **77** 184514
- [24] Huh Y and Sachdev S 2008 *Phys. Rev. B* **78** 064512
- [25] Xu C, Qi Y and Sachdev S 2008 *Phys. Rev. B* **78** 134507
- [26] Fritz L and Sachdev S 2009 *Phys. Rev. B* **80** 144503
- [27] Wang J, Liu G-Z and Kleinert H 2011 *Phys. Rev. B* **83** 214503
- [28] Liu G-Z, Wang J-R and Wang J 2012 *Phys. Rev. B* **85** 174525
- [29] Wang J-R and Liu G-Z 2013 *New J. Phys.* **15** 063007
- [30] Wang J and Liu G-Z 2013 *New J. Phys.* **15** 073039
- [31] She J-H, Lawler M J and Kim E-A 2015 *Phys. Rev. B* **92** 035112
- [32] Pomeranchuk I J 1958 *Sov. Phys. JETP* **8** 361
- [33] Yamase H and Kohno H 2000 *J. Phys. Soc. Jpn.* **69** 332
- [34] Yamase H and Kohno H 2000 *J. Phys. Soc. Jpn.* **69** 2151
- [35] Halboth C J and Metzner W 2000 *Phys. Rev. Lett.* **85** 5162
- [36] Oganessian V, Kivelson S A and Fradkin E 2001 *Phys. Rev. B* **64** 195109
- [37] Hubbard J 1963 *Proc. R. Soc. A* **276** 238
- [38] Anderson P W 1987 *Science* **235** 1196
- [39] Zanchi D and Schulz H J 1996 *Phys. Rev. B* **54** 9509
- [40] Maier T, Jarrell M, Schulthess T C, Kent P R C and White J B 2005 *Phys. Rev. Lett.* **95** 237001
- [41] Gull E, Parcollet O and Millis A J 2013 *Phys. Rev. Lett.* **110** 216405
- [42] Huscroft C, Jarrell M, Maier Th, Moukouri S and Tahvildarzadeh A N 2001 *Phys. Rev. Lett.* **86** 139
- [43] Macridin A, Jarrell M, Maier T, Kent P R C and D'Azevedo E 2006 *Phys. Rev. Lett.* **97** 036401
- [44] Gull E, Ferrero M, Parcollet O, Georges A and Millis A J 2010 *Phys. Rev. B* **82** 155101
- [45] Fang K, Fernando G W and Kocharian A N 2013 *J. Phys.: Condens. Matter* **25** 205601
- [46] Fang K, Fernando G W, Balatsky A V and Kocharian A N 2015 *Phys. Lett. A* **379** 2230
- [47] Nersesyan A A, Tsvelik A M and Wenger F 1994 *Phys. Rev. Lett.* **72** 2628
- [48] Nersesyan A A, Tsvelik A M and Wenger F 1995 *Nucl. Phys. B* **438** 561
- [49] Altland A, Simons B D and Zirnbauer M R 2002 *Phys. Rep.* **359** 283
- [50] Giuliani G F and Vignale G 2005 *Quantum Theory of the Electron Liquid* (Cambridge: Cambridge University Press) ch 8
- [51] Varma C M, Littlewood P B, Schmitt-Rink S, Abrahams E and Ruckenstein A E 1989 *Phys. Rev. Lett.* **63** 1996
- [52] Gonzalez J, Guinea F and Vozmediano M A H 1996 *Phys. Rev. Lett.* **77** 3589
- [53] Gonzalez J, Guinea F and Vozmediano M A H 1999 *Phys. Rev. B* **59** R2474
- [54] Kotov V N, Uchoa B, Pereira V M, Guinea F and Castro Neto A H 2012 *Rev. Mod. Phys.* **84** 1067

- [55] Wang J-R and Liu G-Z 2014 *Phys. Rev. B* **89** 195404
- [56] Hofmann J, Barnes E and Das Sarma S 2014 *Phys. Rev. Lett.* **113** 105502
- [57] Chiao M, Hill R W, Lupien C, Taillefer L, Lambert P, Gagnon R and Fournier P 2000 *Phys. Rev. B* **62** 3554
- [58] Kopnin N B and Sonin E B 2008 *Phys. Rev. Lett.* **100** 246808
- [59] Nandkishore R, Maciejko J, Huse D A and Sondhi S L 2013 *Phys. Rev. B* **87** 174511
- [60] Sur S and Lee S-S 2015 *Phys. Rev. B* **91** 125136
- [61] Goswami P and Chakravarty S 2011 *Phys. Rev. Lett.* **107** 196803
- [62] Roy B and Das Sarma S 2014 *Phys. Rev. B* **90** 241112(R)
- [63] Shankar R 1994 *Rev. Mod. Phys.* **66** 129
- [64] Metlitski M A, Mross D F, Sachdev S and Senthil T 2015 *Phys. Rev. B* **91** 115111
- [65] Fitzpatrick A L, Kachru S, Kaplan J, Raghu S, Torroba G and Wang H 2015 *Phys. Rev. B* **92** 045118
- [66] Balatsky A V, Vekhter I and Zhu J-X 2006 *Rev. Mod. Phys.* **78** 373
- [67] Fisher R A, Gordon J E and Phillips N E 2007 *Handbook of High-Temperature superconductivity* ed J R Schrieffer and J S Brooks (Berlin: Springer) ch 9
- [68] Revaz B, Genoud J-Y, Junod A, Neumaier K, Erb A and Walker E 1998 *Phys. Rev. Lett.* **80** 3364
- [69] Wright D A, Emerson J P, Woodfield B F, Gordon J E, Fisher R A and Phillips N E 1999 *Phys. Rev. Lett.* **82** 1550
- [70] Riggs S C, Vafek O, Kemper J B, Betts J B, Migliori A, Balakirev F F, Hardy W N, Liang R, Bonn D A and Boeinger G S 2011 *Nat. Phys.* **7** 332
- [71] Allais A and Senthil T 2012 *Phys. Rev. B* **86** 045118
- [72] Kivelson S A and Varma C M arXiv:1208.6498v1
- [73] Wang L and Vafek O 2013 *Phys. Rev. B* **88** 024506
- [74] Berg E, Chen C-C and Kivelson S A 2008 *Phys. Rev. Lett.* **100** 027003
- [75] Shekhter A, Ramshaw B J, Liang R, Hardy W N, Bonn D A, Balakirev F F, McDonald R D, Betts J B, Riggs S C and Migliori A 2013 *Nature* **498** 75
- [76] Zaanen J 2013 *Nature* **498** 41
- [77] Valla T, Fedorov A V, Johnson P D, Wells B O, Hulbert S L, Li Q, Gu G D and Koshizuka N 1999 *Science* **285** 2110
- [78] Titov M L, Yashenkin A G and Aristov D N 1995 *Phys. Rev. B* **52** 10626
- [79] Khveshchenko D V and Paaske J 2001 *Phys. Rev. Lett.* **86** 4672
- [80] Plumb N C, Reber T J, Koralek J D, Sun Z, Douglas J F, Aiura Y, Oka K, Eisaki H and Dessau D S 2010 *Phys. Rev. Lett.* **105** 046402
- [81] Wang J and Liu G-Z 2015 *Phys. Rev. B* **92** 184510
- [82] Metzger W, Rohe D and Andergassen S 2003 *Phys. Rev. Lett.* **91** 066402
- [83] Dell'Anna Luca and Metzner W 2006 *Phys. Rev. B* **73** 045127
- [84] Rech J, Pépin C and Chubukov A V 2006 *Phys. Rev. B* **74** 195126
- [85] Garst M and Chubukov A V 2010 *Phys. Rev. B* **81** 235105
- [86] Metlitski M A and Sachdev S 2010 *Phys. Rev. B* **82** 075127
- [87] Isobe H, Yang B-J, Chubukov A, Schmalian J and Nagaosa N 2016 *Phys. Rev. Lett.* **116** 076803
- [88] Cho G Y and Moon E-G 2016 *Sci. Rep.* **6** 19198
- [89] Pelissetto A P, Sachdev S and Vicari E 2008 *Phys. Rev. Lett.* **101** 027005
- [90] Chubukov A and Hirschfeld P J 2015 *Phys. Today* **68** 46
- [91] Fernandes F M, Chubukov A V and Schmalian J 2014 *Nat. Phys.* **10** 97
- [92] Böhmer A E, Arai T, Hardy F, Hattori T, Iye T, Wolf T, Löhneysen H v, Ishida K and Meingast C 2015 *Phys. Rev. Lett.* **114** 027001
- [93] Baek S-H, Efremov D V, Ok J M, Kim J S, Brink J v d and Büchner B 2015 *Nat. Mater.* **14** 210
- [94] Chubukov A V, Fernandes R M and Schmalian J 2015 *Phys. Rev. B* **91** 201105(R)
- [95] Yu R and Si Q 2015 *Phys. Rev. Lett.* **115** 116401
- [96] Glasbrenner J K, Mazin I I, Jeschke H O, Hirschfeld P J, Fernandes R M and Valenti R 2015 *Nat. Phys.* **11** 953
- [97] Wang F, Kivelson S A and Lee D-H 2015 *Nat. Phys.* **11** 959
- [98] Jiang K, Hu J, Ding H and Wang Z 2016 *Phys. Rev. B* **93** 115138
- [99] Lee C-H, Yin W-G and Ku W 2009 *Phys. Rev. Lett.* **103** 267001
- [100] Shibauchi T, Carrington A and Matsuda Y 2014 *Annu. Rev. Condens. Matter Phys.* **5** 113
- [101] Dioguardi A P, Kissikov T, Lin C H, Shirer K R, Lawson M M, Grafe H-J, Chu J-H, Fisher I R, Fernandes R M and Curro N J 2016 *Phys. Rev. Lett.* **116** 107202
- [102] Fernandes R M and Schmalian J 2012 *Supercond. Sci. Technol.* **25** 084005