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Chaotic Motion in a Harmonically Excited Soliton System*

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Abstract *The influence of a soliton system under an external harmonic excitation is considered. We take the compound KdV-Burgers equation as an example, and investigate numerically the chaotic behavior of the system with a periodic forcing. Different routes to chaos such as period doubling, quasi-periodic routes, and the shapes of strange attractors are observed by using bifurcation diagrams, the largest Lyapunov exponents, phase projections and Poincaré maps.*

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Key words: soliton system, compound KdV-Burgers equation, chaos

1 Introduction

The soliton systems such as KdV equation, Burgers equation and nonlinear Schrödinger equation have received great attention during the last decades.[1−7] These nonlinear evolution equations arise from many physical fields, and many of them are completely integrable. However there often exist various perturbations for a practical physical process. The nature of this external perturbation can be different and varies from one physical problem to another. So the growing interest in the study of soliton equations under external perturbations has recently more developed.[8−11]

As is well known, a completely integrable soliton equation cannot display chaotic behavior. But the addition of a perturbation to an integrable equation may lead to chaotic dynamics. For example, chaos has been found in the perturbed sine-Gordon equation,[12] and in the cubic nonlinear Schrödinger equation.[13] For the perturbed KdV equation, the conditions of their chaotic behavior are studied with the help of the Melnikov theory.[14]

In this work we start with an important soliton equation, say the compound KdV-Burgers equation,

$$u_t + auu_x + bu^2u_x + ru_{xx} + su_{xxx} = 0\,, \qquad (1)$$

where r is a damping parameter, and a, b, and s are real parameters. Equation (1) is applied as a model for long-wave propagation in nonlinear media with dispersion and dissipation,[15] and has been studied by many authors.[15,16] It is also known to possess travelling wave solutions in the form of solitary or kink wave solutions.

2 Dynamical Model

We consider a parametric excitation for the compound KdV-Burgers equation, that is,

$$u_t + auu_x + bu^2u_x + ru_{xx} + su_{xxx} = f_x\,, \qquad (2)$$

where f is the driving term.

Let $u(x,t) = \varphi(\xi)$ with $\xi = x - vt$ being a travelling wave solution for Eq. (2). Substituting $\varphi(\xi)$ into Eq. (2) yields

$$-v\varphi_\xi + a\varphi\varphi_\xi + b\varphi^2\varphi_\xi + r\varphi_{\xi\xi} + s\varphi_{\xi\xi\xi} = f_\xi\,, \qquad (3)$$

where $\varphi_\xi = \mathrm{d}\varphi/\mathrm{d}\xi$. To simplify the analysis, we assume that f is a periodic function. Taking $f = f_0\cos(\omega\xi)$ and integrating Eq. (3) once leads to

$$\varphi_{\xi\xi} + \mu\varphi_\xi - \alpha\varphi + \beta\varphi^2 + \delta\varphi^3 = g\cos(\omega\xi)\,, \qquad (4)$$

where $\mu = r/s$, $\alpha = v/s$, $\beta = a/(2s)$, $\delta = b/(3s)$, and $g = f_0/s$.

Equation (4) is the asymmetric double-well Helmholtz–Duffing oscillator with quadratic and cubic nonlinearities. It describes the dynamic finite behavior of prestressed elastic structures, (cables, arches) subjected to deterministic harmonic forcing in the vertical plane. As a mechanical model with single degree of freedom, equation (4) with α, β and $\delta > 0$ is applied in practical engineering extensively. For instance, it may describe the vibration of plate spring in rare earth giant magnetostriction transducer.

In addition, equation (4) is also used as the biomathematical model of the aneurysm of circle of Willis when $\delta > 0$, α and $\beta < 0$, and it simulates the blood flow inside aneurysm.[17,18] Here the variable φ in Eq. (4) represents the velocity of blood flow, and $g\cos(\omega\xi)$ indicates the rate of change of the central blood pressure, where g is equivalent to the pulse pressure and ω is inverse of the cardiac frequency. Contrary to the Duffing equation

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(Eq. (4) with $\beta = 0$), which was studied extensively from a theoretical, numerical, experimental, and control point of view,[19,20] equation (4) was not deeply investigated in the past.[21]

The equivalent form of the perturbed system (4) can be written as ($X = \varphi$)

$$\dot{X} = Y\,, \quad \dot{Y} = g\cos(\omega\xi) - \mu Y + \alpha X - \beta X^2 - \delta X^3\,, \quad (5)$$

where dot denotes the derivative with respect to ξ.

The undamped and unforcing system of Eq. (5) (with $\mu = 0$, f=0) is an integrable Hamiltonian system with a Hamiltonian function,

$$H(X,Y) = \frac{1}{2}Y^2 - \frac{\alpha}{2}X^2 + \frac{\beta}{3}X^3 + \frac{\delta}{4}X^4\,, \quad (6)$$

the corresponding potential is a ($\phi^3 + \phi^4$)-model, given by

$$V(X) = -\frac{\alpha}{2}X^2 + \frac{\beta}{3}X^3 + \frac{\delta}{4}X^4\,. \quad (7)$$

3 Chaos in Perturbed System

In order to obtain some information of chaotic behavior for the perturbed system (5), several different numerical tools that effectively describe the bifurcation and chaotic phenomenon are applied, which are (i) bifurcation diagrams, (ii) the largest Lyapunov exponents, (iii) the Poincaré map and phase plane plots, and (iv) the fractal dimension. These methods are all very useful tools for examining chaotic properties of dynamic systems and exploring chaotic attractors.

System (5) involves six parameters: g, μ, α, β, δ, and ω. To simplify the analysis, we only choose ω as the control parameter.

We first fix $g = 0.25$, $\mu = 0.2$, $\alpha = 0.5$, $\beta = 0.4$ and $\delta = 0.5$, and let α to change in a wide range. The result of a numerical investigation of the model is reported. Figure 1 shows a one-dimensional bifurcation diagram and the corresponding largest Lyapunov exponent. From Fig. 1(a), we can see that the nonlinear dynamical system (5) exhibits periodic and chaotic behaviors when the parameter ω varies. The largest Lyapunov exponent given by Fig. 1(b) can be convinced of occurrence of chaotic motion. Figure 2 shows some local amplification of Fig. 1(a). It can be also observed, from Figs. 1 and 2, that the system enters chaotic states usually through a sequence of period doubling, while period doubling is at present the most commonly known route to chaos.

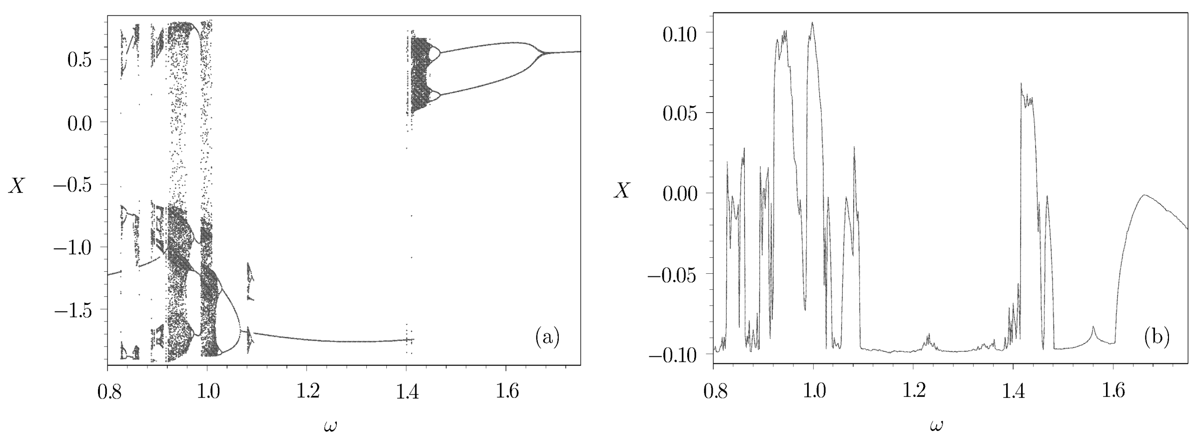

Fig. 1 (a) Bifurcation diagram in (ω–X) plane for Eq. (5) with $g = 0.25$, $\mu = 0.2$, $\alpha = 0.5$, $\beta = 0.4$ and $\delta = 0.5$; (b) The largest Lyapnouv exponent corresponding to (a).

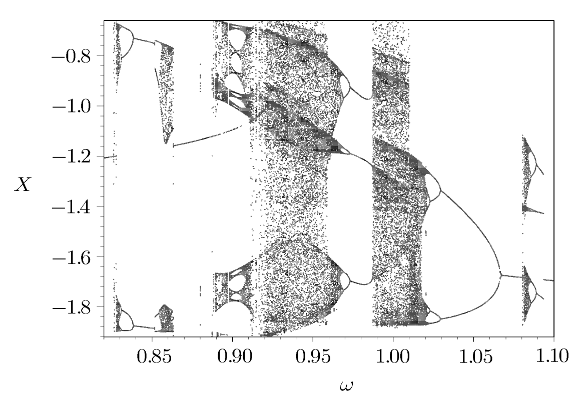

Fig. 2 A local amplification of Fig. 1(a).

Figure 3 describes the Phase projection of a chaotic state in Fig. 1(a) with $\omega = 0.94$. The largest Lyapunov exponent is found to be positive: $\lambda = 0.101$, and to indicate chaotic dynamics. The Lyapunov dimension can be obtained, according to the definition given by Kaplan and Yorke.[22] The calculated fractal dimension of the above strange attractor is 2.334.

Next we fix $g = 0.8$, $\mu = 0.6$, $\alpha = 0.3$, $\beta = -0.8$ and $\delta = 1.5$, and let the perturbed frequency ω to vary in some range. When ω is changed from 0.8 to 1.6, the dynamical system also exhibits various periodic and chaotic behaviors. The bifurcation diagram and the corresponding largest Lyapunov exponent are shown in Fig. 4.

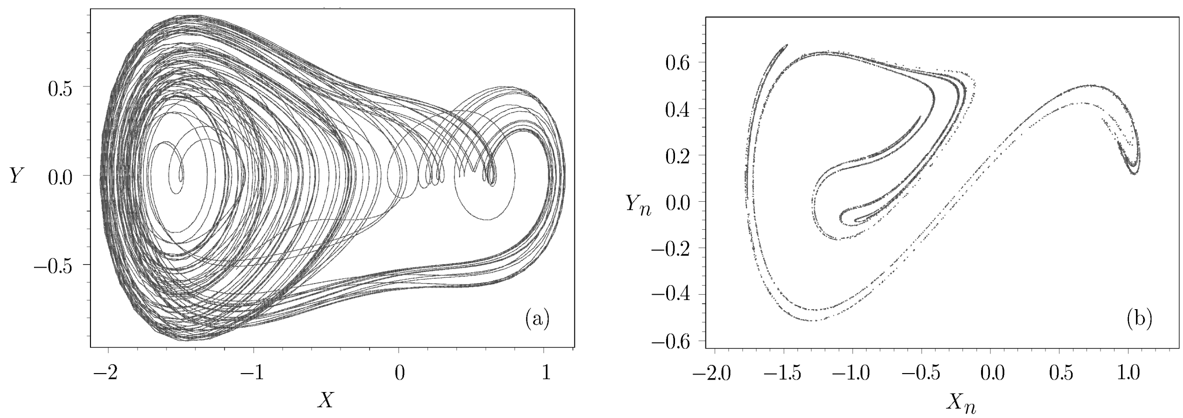

Fig. 3 (a) Phase projection of a chaotic state in Fig. 1(a) with $\omega = 0.94$; (b) The Poincaré map corresponding to (a).

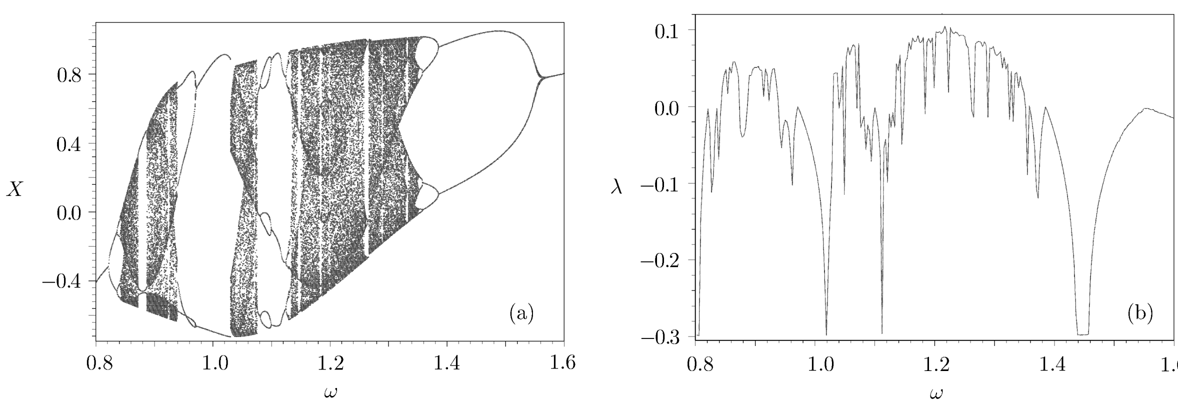

Fig. 4 (a) Bifurcation diagram in (ω–X) plane for Eq. (5) with $g = 0.8$, $\mu = 0.6$, $\alpha = 0.3$, $\beta = -0.8$, and $\delta = 1.5$; (b) The largest Lyapnouv exponent corresponding to (a).

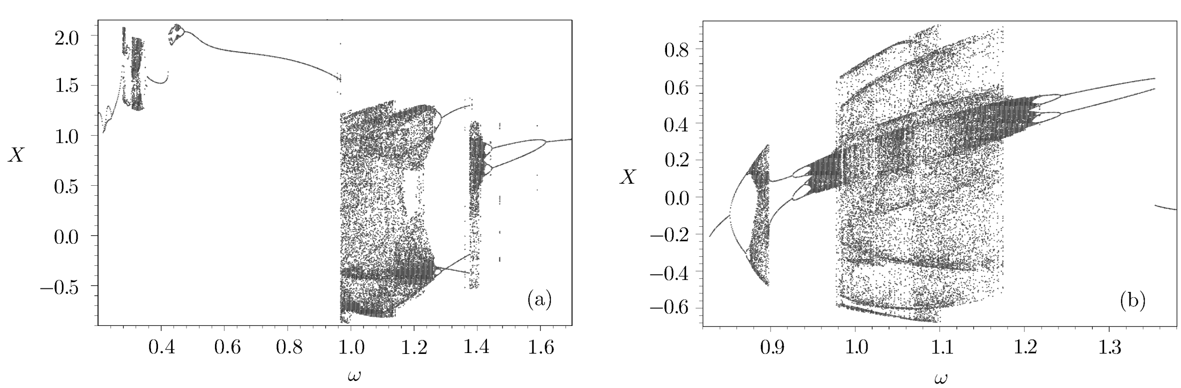

Fig. 5 Bifurcation diagram in (ω–X) plane for Eq. (5) with (a) $g = 0.386$, $\mu = 0.16$, $\alpha = -0.1$, $\beta = -0.66$, and $\delta = 0.5$; (b) $g = 0.45$, $\mu = 0.2$, $\alpha = -0.2$, $\beta = -0.8$, and $\delta = 0.6$.

When we take g, μ, α, β, δ as 0.386, 0.16, −0.1, −0.66, 0.5 and 0.45, 0.2, −0.2, −0.8, 0.6, respectively, the dynamical responses of system (5) varying with ω are described in Fig. 5. As shown in Fig. 5, the possible chaotic solutions of system (5) versus different bifurcation parameter ω are obtained clearly. To give some better insight of chaos feature, we will display some strange attractors in the Poincaré maps. Figure 6 depicts the phase trajectory and Poincaré maps of both chaotic states in Fig. 5(a) with $\omega = 1.05$ and 1.405. The largest Lyapnouv exponents corresponding to Figs. 6(a) and 6(b) and Figs. 6(c) and 6(d) are 0.104 and 0.080, respectively. And the Lyapnouv dimensions are 2.398 and 2.328, respectively.

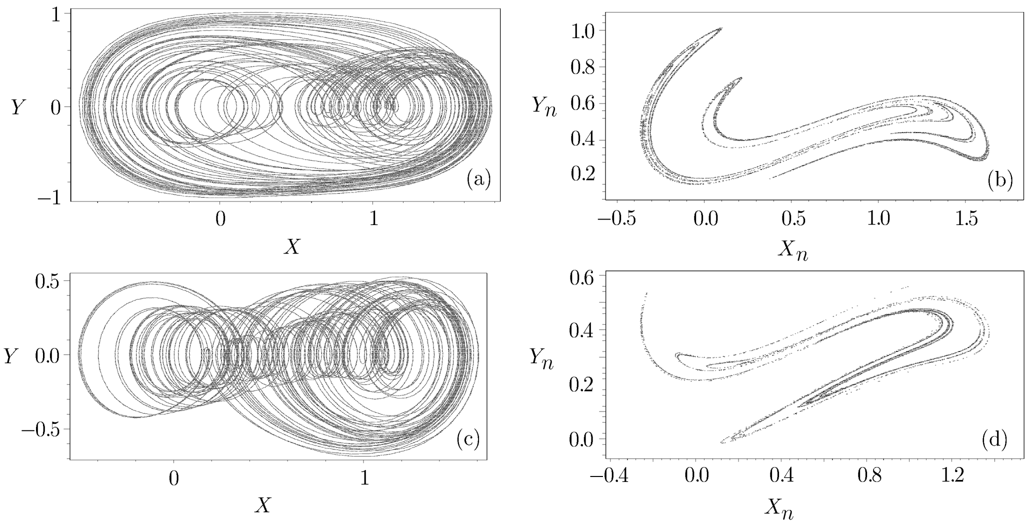

Fig. 6 Phase projections and the Poincaré maps of some chaotic states in Fig. 5(a), with (a) (b) $\omega = 1.05$; (c) (d) $\omega = 1.405$.

4 Conclusion

In this paper the bifurcation and chaotic motion of the compound KdV-Burgers equation under a harmonic excitation are studied numerically. It is shown that the dynamical chaos can occur when appropriately choose the system parameters and initial conditions. Different routes to chaos such as period doubling, quasi-periodic routes, and the shapes of strange attractors are observed by applying bifurcation diagrams, phase projections and Poincaré maps. To characterize chaotic behavior of the perturbed system (5), the spectrum of Lyapunov exponents and Lyapunov dimensions are also employed.

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